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MATHEMATICS

ON THE SECONDARY SPECTRUM OF HYDROGEN.

BY

PANCHANON DAS, M.Sc.,

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Although the application by Bohr and Sommerfeld, of Quantum-theory to spectroscopy, has met with a phenomenal success, the only atoms, of which the line-spectra, have as yet been quantitatively accounted for are those of hydrogen and ionised helium. The next-structure in the order of complexity is the helium atom. But as it is a problem of three bodies, no exact solutions have been obtained. The hydrogen molecule is still more complex, as it consists of two electrons and two hydrogen-nuclei. The only existing model of the hydrogen-molecule is that of Debye, but the dynamical solution of the Debye-model has not been effected yet, although it has attracted the attention of Silberstein and Saha. It is practically established now that the secondary spectrum of hydrogen is emitted by the hydrogen-molecule. So a theory of the secondary spectrum must rest on a workable model of the molecule. The present paper embodies such a model and the Hamilton-Jacobian equation of the same can be solved to a certain order of approximation. A frequency-formula has been deduced; but as the secondary spectrum of hydrogen consists of an extremely large number of lines, it is idle to identify each of these lines with some lines calculated from formula, at the present state of the subject. But as Sommerfeld has pointed out, some other features of the secondary spectrum should be studied with a view to observing some regularity, and if such is observed, any new theory should be based on them. Glitscher has observed¹ that if the maxima in the energy-diagram of the secondary spectrum are superposed on Balmer lines, then each Balmer line nearly coincides with a corresponding maximum. The model suggested here appears to account for this to some extent.

In order to arrive at the proposed model we start from the genesis of the molecule. When two neutral hydrogen atoms approach each other, the centre of gravity of their nuclei is approximately at rest

or in uniform motion, since the masses of the electrons are small compared with those of the nuclei. Now since the mutual force between the nuclei is one of repulsion they would fly apart after some time unless some electrons intervene. Let us place one electron at the centre of gravity of these two nuclei, so that this electron moves in a practically force-free field. The two nuclei will now describe closed orbits about this electron, and these would form an ellipse-verein or degenerate into a circle. The remaining electron may now describe some orbit at a comparatively large distance from this complex structure. This completes the model.

The potential of this complex structure consisting of an electron and two nuclei can be shown to be approximately that of two centres¹ of force. Or we may replace the two revolving nuclei by a ring of electric charge and expand its potential in a series form. But it is immaterial which way we regard the problem, as it is the form of the series-formula that concerns us and not the numerical value of the constants involved therein. In accordance with the latter view the potential is of the form,

$$V = \frac{e''}{r} + \frac{e_1^2}{r^2} + \frac{e_2^2}{r^2} + \dots \quad 1$$

If we retain terms up to $1/r^2$ only and quantise the atom in the polar coordinates of the valency-electron, we shall then arrive at the usual Rydberg Formula :—

$$\gamma = N \left| \frac{1}{(n+\alpha)^2} - \frac{1}{(\rho+\beta)^2} \right|$$

Now the term e_1^2 is proportional to the square of the radius of the ring of nuclei. If we assume that this nuclear ring also is quantised, then e_1 becomes a function of these quantum numbers. Thus the terms α, β in Rydberg's formula quoted above, are also functions of these quantum-numbers. Assuming that these nuclei, each of mass M , describe the same circle with radius a , and have an angular velocity ω , the equation of motion is

$$Ma\omega^2 = \frac{3e^2}{4a^2}$$

¹ Cf. Taub. Ann. d. Physik, 1919.

subject to the quantum-condition,

$$2Ma^2\omega = \frac{n_a h}{2\pi},$$

Hence
$$a = \frac{n_a^2 h^2}{12\pi^2 M e^2}$$

It is shown in Sommerfeld's *Atom-bau*, 2nd *ed.* that the constant a in Rydberg's formula is proportional to e_1 and therefore to n_a^2 .

We now introduce the idea that when a quantum-radiation takes place, both the numbers n and n_a suffer a quantum-transit, which amounts to saying that a re-arrangement of both the inner structure and the outer electrons takes place during a radiation. Under the circumstances, the frequency-formula takes the form.

$$\gamma = \frac{N_a}{(n+k.n_a^{-1})^2} - \frac{N_a}{(p+k.p_a^{-1})^2}$$

where k is a small quantity of the order of u/M .

This can be approximately written as follows:—

$$\gamma = N_a \left(\frac{1}{n^2} - \frac{1}{p^2} \right) - N_a . 2k . \left(\frac{n_a^{-1}}{n^3} - \frac{p_a^{-1}}{p^3} \right).$$

From the expression for the radius of the nuclear ring, it is evident that unless n_a is very large, this ring is much smaller than the one-quantum or two-quantum orbit of the outer electron. It is thus obvious that corresponding to every quantum-jump of the outer electron, quite a large number of transitions in the value of n_a is possible, and this will give rise to a large number of closely grouped lines. This accounts for the many-lined character of the secondary spectrum. Also the form of the frequency-formula at once visualises a close relation between the secondary-spectrum and the Balmer lines. For instance, if we put $n=2$ and $p=3$, this corresponds to a quantum-transit in the hydrogen atom which gives rise to the line $H\alpha$. In the molecule however we shall get large cluster of lines somewhere near $H\alpha$. Similarly, if we put $n=2$ and $p=1$, we shall get another cluster of lines in the neighbourhood of $H\beta$ and so on. It is to be borne in mind that these clusters most probably overlap each other so that no well-defined line of demarcation exists between them. The net appearance of this

theoretical spectrum is neither that of bands nor of series-lines. The puzzling character of the secondary spectrum and its non-conformity to any class is quite well-known.

Of course an actual calculation of these lines and their identification with the individual lines in the secondary spectrum will lead to nowhere, but there are other features of the formula which lend themselves easily to an experimental test.

As already mentioned in the introduction, Glitscher observed that if the maxima in the energy-diagram of the secondary spectrum of hydrogen were denoted by the symbol H'_α , H'_β , etc., then the difference in wave-number between the above and the Balmer-lines; *i.e.*, the quantities $H'_\alpha - H_\alpha$, $H'_\beta - H_\beta$, etc., were approximately constant.

We can reasonably assume that the state of the inner core of the molecule which corresponds to these maxima, must be the most probable states. Let $n_a = p_a$ be the quantum-number denoting these most probable states. Then it is easily seen that

$$H'_\alpha - H_\alpha = -N_a \cdot 2k \cdot n_a^{-1} \left(\frac{1}{2^3} - \frac{1}{3^3} \right)$$

$$H'_\beta - H_\beta = -N_a \cdot 2k \cdot n_a^{-1} \left(\frac{1}{2^3} - \frac{1}{4^3} \right)$$

$$H'_\gamma - H_\gamma = -N_a \cdot 2k \cdot n_a^{-1} \left(\frac{1}{2^3} - \frac{1}{5^3} \right), \text{ etc.}$$

Hence the ratios of these are

$$= \frac{1}{2^3} - \frac{1}{3^3} : \frac{1}{2^3} - \frac{1}{4^3} : \frac{1}{2^3} - \frac{1}{5^3} :$$

$$= 0.73 : 0.91 : 0.97 : 1 :$$

On the other hand from the energy-diagram we find :—

$$H'_\alpha - H_\alpha : H'_\beta - H_\beta : H'_\gamma - H_\gamma :$$

$$= 1380 : 1470 : 1542 : 1652 : \dots = 0.84 : 0.89 : 0.95 : 1 :$$

If the agreement between the theoretical and observed values is not quite close, these values are at least of the same order of magnitude. We must make room for the possibility that n_a may not equal p_a in a

quantum jump corresponding to any maximum in the energy-diagram but may be slightly different from one another: this will certainly modify the value of the ratios to some extent.

It is natural to expect that the complex core of the molecule is not very stable. It is thus obvious that at low voltages, the spark or the arc spectrum should consist of the secondary lines emitted by the molecule: and when the voltage is increased, the molecule atomises and the Balmer lines should make their appearance. This is well corroborated by experimental results.

There are reasons to believe that the resonance and ionisation-potentials determined by electron-collision (Horton and Davies) are not really those of the atom, but of the molecule (see Foote and Mohler's "Origin of Spectra," p. 75). If we put $n=1$ instead $n=2$, as in Lyman's series, then the ionisation and resonance-potentials of our molecule will differ from those of the atom by a quantity of the order m/M . Hence our model bears out the conjecture of Foote and Mohler in a striking manner.

HIGHER ORDER TIDES IN CANALS OF VARIABLE SECTION.

By

NRIPENDRA NATH SEN, D.Sc.

The only problem of higher order tides completely solved is the one considered by Airy¹ and McCowan² in which the section is uniformly rectangular. The object of the present paper is (1) to establish the exact equations for free tidal oscillations in canals of variable section and (2) to determine higher order tides in a parabolic canal.³

2. Taking the origin on the undisturbed level and the axis of x parallel to the length of the canal, the exact equations for free tidal oscillations in canals of variable section may easily be seen to be

$$\frac{\partial \eta}{\partial t} + \frac{1}{b} \frac{\partial}{\partial x} b (h + \eta) u = 0 \quad \dots (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} = -g \frac{\partial \eta}{\partial x} \quad \dots (2)$$

where u , η , b , h , p are velocity, tidal elevation, breadth, depth and pressure at a distance x .

3. Let $b = b_0$, $h = h_0 \left(1 - \frac{x^2}{a^2}\right)$, so that the longitudinal section is a parabola. In this case, we have from (1) and (2), after a little simplification,

$$\begin{aligned} \frac{\partial^2 \eta}{\partial t^2} - \frac{gh_0}{a^2} \frac{\partial}{\partial z} (1 - z^2) \frac{\partial \eta}{\partial z} = -\frac{1}{u} \frac{\partial}{\partial z} \frac{\partial}{\partial t} (u\eta) \\ + \frac{h_0}{a^2} \frac{\partial}{\partial z} (1 - z^2) u \frac{\partial u}{\partial z} \end{aligned} \quad (3)$$

¹ Airy—"Tides and Waves" Ency. Metrop. Art., 192, 1845.

² McCowan—"On the theory of long waves, etc." Phil. Mag., Series 5, Vol. 35, 250, 1898.

Also Lamb—"Hydrodynamics," Ed. IV, p. 251 and p. 273, (1916).

³ This problem was attempted in a previous issue of this bulletin (Vol. X, 113, 1918-19) but the solution obtained therein is wrong due to the use of incorrect equations of motion.

where

$$z = \frac{x}{a}$$

Neglecting squares and higher powers of u and η , we have from (2) and (3)

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{gh_0}{a^2} \frac{\partial}{\partial z} (1-z^2) \frac{\partial \eta}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x}$$

Assuming that $\eta \propto e^{i\sigma t}$ and putting

$$\sigma^2 = u(u+1) \frac{gh_0}{a^2} \quad \dots (4)$$

we have

$$\frac{\partial}{\partial z} (1-z^2) \frac{\partial \eta}{\partial z} + u(u+1) \eta = 0$$

whence

$$\eta = C P_n(z) e^{i\sigma t}$$

where n is to be integral since η must be finite when $x = \pm a$, i.e. $z = \pm 1$;

$$\therefore \text{ From (2), } u = \frac{igC}{a\sigma} P'_n(z) e^{i\sigma t}$$

correct to above order of approximation.

Now, substituting the above values of u and η in the 2nd order terms in (3) and neglecting third and higher powers of u , η and their differential co-efficients, we have after a little simplification.

$$\begin{aligned} \frac{\partial}{\partial z} (1-z^2) - \frac{u(u+1)}{a^2} \frac{\partial^2 \eta}{\partial t^2} \\ = \frac{gC^2}{a^2 \sigma^2} \frac{\partial}{\partial z} [2z P'_n{}^2(z) - 3n(n+1) P_n(z) P'_n(z)] e^{2i\sigma t} \dots (5) \end{aligned}$$

Assuming

$$\eta = C P_n(z) e^{i\sigma t} + \sum k_r P_r(z) e^{2i\sigma t} \quad \dots (6)$$

where k_r is so small that its second and higher powers can be neglected, we have from (5) and (6)

$$\begin{aligned} & \sum k_r [4n(n+1) - r(r+1)] P_r(z) \\ &= \frac{gC^2}{a^2\sigma^2} \frac{\partial}{\partial z} [2z P_n'^2(z) - 3n(n+1) P_n(z) P_n'(z)] \dots \quad (7) \end{aligned}$$

Now, it can be easily proved that

$$\begin{aligned} P_n'(z) &= (2n-1) P_{n-1}(z) + (2n-5) P_{n-3}(z) \\ &\quad + (2n-9) P_{n-5}(z) + \dots \quad (8)^1 \end{aligned}$$

$$\begin{aligned} z P_n'(z) &= n P_n(z) + (2n-3) P_{n-2}(z) \\ &\quad + (2n-7) P_{n-4}(z) + \dots \quad (9)^2 \end{aligned}$$

$$\begin{aligned} \text{Also } P_m(z) P_n(z) &= \sum_{r=0}^m \frac{\Lambda_{m-r} \Lambda_r \Lambda_{n-r}}{\Lambda_{m+n-r}} \\ &\quad \cdot \frac{2(n+m)+1-4r}{2(n+m)+1-2r} P_{n+m-2r}(z) \dots \quad (10)^3 \end{aligned}$$

where

$$\Lambda_m = \frac{2m}{m+1} \quad \text{and} \quad \Lambda_m = \frac{2m}{m-1} \quad \dots \quad (11)$$

$$\begin{aligned} \therefore 2z P_n'^2(z) - 3n(n+1) P_n(z) P_n'(z) &= P_n'(z) [2z P_n'(z) - 3n(n+1) P_n(z)] \\ &= [(2n-1) P_{n-1}(z) + (2n-5) P_{n-3}(z) + (2n-9) P_{n-5}(z) + \dots] \\ &\quad [-n(3n+1) P_n(z) + 2(2n-3) P_{n-2}(z) + 2(2n-7) P_{n-4}(z) + \dots] \\ &= [B_{2n-1} P_{2n-1}(z) + B_{2n-3} P_{2n-3}(z) + \dots] \dots \quad (12) \end{aligned}$$

expressing the product of two Legendre's co-efficients in terms of Legendre's co-efficients by (10).

¹ Whittaker—Mod. Analysis, p. 303, Result IV.

² Whittaker, *ibid*, p. 324, Ex. 4.

³ Adams—Proc. Roy. Soc., Vol. 27. Also Whittaker, *ibid*, p. 325.

Now, it may be easily seen that

$$\begin{aligned}
 B_{2n-1} &= -n(2n-1)(3n+1) \frac{A_n A_{n-1}}{A_{2n-1}} \\
 B_{2n-3} &= -n(2n-1)(3n+1) \cdot \frac{4n-5}{4n-3} \cdot \frac{A_{n-1} A_1 A_{n-2}}{A_{2n-2}} \\
 &\quad + 2(2n-1)(2n-3) \frac{A_{n-1} A_{n-2}}{A_{2n-3}} - n(2n-5)(3n+1) \frac{A_n A_{n-3}}{A_{2n-3}} \\
 B_{2n-5} &= -n(2n-1)(3n+1) \frac{4n-9}{4n-5} \frac{A_{n-2} A_2 A_{n-3}}{A_{2n-3}} \\
 &\quad + 2(2n-1)(2n-3) \frac{4n-9}{4n-7} \frac{A_{n-2} A_1 A_{n-3}}{A_{2n-4}} \\
 &\quad - n(2n-5)(3n+1) \frac{4n-9}{4n-7} \frac{A_{n-1} A_1 A_2}{A_{2n-4}} \\
 &\quad + 2(2n-1)(2n-7) \frac{A_{n-1} A_{n-4}}{A_{2n-5}} \\
 &\quad + 2(2n-3)(2n-5) \frac{A_{n-2} A_{n-3}}{A_{2n-5}} - n(3n+1)(2n-9) \frac{A_n A_{n-5}}{A_{2n-5}}
 \end{aligned}$$

etc., etc., where A_m is given by (11).

Again by (8), we have

$$\begin{aligned}
 \frac{d}{dz} [B_{2n-1} P_{2n-1}(z) + B_{2n-3} P_{2n-3}(z) + B_{2n-5} P_{2n-5}(z) + \dots] \\
 = C_{2n-2} P_{2n-2}(z) + C_{2n-4} P_{2n-4}(z) + C_{2n-6} P_{2n-6}(z) + \dots
 \end{aligned}$$

where

$$\begin{aligned}
 C_{2n-2} &= (4n-3) B_{2n-1} \\
 C_{2n-4} &= (4n-7) [B_{2n-1} + B_{2n-3}] \\
 C_{2n-2j} &= (4n-4j+1) \sum_{r=1}^j B_{2n-2r+1} \dots \quad (13)
 \end{aligned}$$

∴ From (7), we have

$$\sum k_r [4n(n+1) - r(r+1)] P_r(z) = \frac{gC^2}{a^2\sigma^2} \sum_{j=1}^n C_{2n-2j} P_{2n-2j}(z)$$

which gives

$$k_{2r+1} = 0 \quad (r=0, 1, 2 \text{ etc.})$$

$$K_{2n-2j} = \frac{gC^2}{a^2\sigma^2} [4n(n+1) - (2n-2j)(2n-2j+1)] C_{2n-2j}$$

where

$$j=1, 2, \dots, n.$$

Hence from (6),

$$\eta = C P_n(z) e^{i\sigma t} + \frac{gC^2}{a^2\sigma^2} \sum_{j=1}^n [4n(n+1) - (2n-2j)(2n-2j+1)] P_{2n-2j}(z) e^{2i\sigma t} \quad \dots (14)$$

where C_{2n-2j} is given by (13).

4. From (14), it is evident that the 2nd order tides are proportional to C^2 and their frequency is double that of the primary disturbance. If the approximation be continued it can be shown that p^{th} order tides are proportional to C^p and its frequency is p times that of the primary.

5. The following particular cases of interest may be easily deduced from the above results

$$(a) \text{ If } n=1, \text{ i.e. } \frac{\sigma^2 a^2}{gh_0} = 1.2, \eta = C P_1(z) e^{i\sigma t} - \frac{gC^2}{a^2\sigma^2} e^{2i\sigma t}$$

$$(b) \text{ If } n=2, \text{ i.e. } \frac{\sigma^2 a^2}{gh_0} = 2.3, \eta = C P_2(z) e^{i\sigma t} - \frac{gC^2}{a^2\sigma^2} \left\{ \frac{3}{2} + 8P_2(z) \right\} e^{2i\sigma t}$$

$$(c) \text{ If } n=3, \text{ i.e. } \frac{\sigma^2 a^2}{gh_0} = 3.4,$$

$$\eta = C P_3(z) e^{i\sigma t} - \frac{gC^2}{a^2\sigma^2} e^{2i\sigma t} \left\{ 3 + \frac{720}{49} P_2(z) + \frac{1125}{49} P_4(z) \right\}$$

(d) If $n=4$, i.e. $\frac{\sigma^2 a^2}{gh_0} = 4.5$,

$$\eta = C P_4(z) e^{-\frac{g}{a^2 \sigma^2} C^2} e^{2i\sigma t} \left\{ 5 + \frac{2750}{111} P_2(z) + \frac{465}{11} P_4(z) + \frac{31850}{627} P_6(z) \right\}$$

ON THE PRODUCT OF BESSEL FUNCTIONS.

BY

K. BASU.

Mr. Abanibhusan Datta (Bull. Cal. Math. Soc., Vol. XII (3), 1921) found out an expression for the product of two Bessel functions in a series of Bessel functions by *two* distinct methods but it was incomplete in as much as he did not lay much stress upon the *coefficients*. The present paper embodies a *third* method of the same, which seems to be an interesting and straight-forward analysis and attempt has been made so that it is applicable to *any number* of Bessel functions. The second section of my treatment involves a method by means of which one can effect indefinite integration of *any number* of products of Bessel functions.

I

Schönholzer established

$$J_{\mu}(z)J_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\mu + \nu + 2n + 1) (\frac{1}{2}z)^{\mu + \nu + 2n}}{n! \Gamma(\mu + n + 1) \Gamma(\nu + n + 1) \Gamma(\mu + \nu + n + 1)},$$

for all values of μ and ν . Also Neumann proved

$$(\frac{1}{2}z)^n = \sum_{r=0}^{\infty} \frac{(n+2r)(n+r-1)!}{r!} J_{n+2r}(z),$$

whence

$$(\frac{1}{2}z)^{\mu + \nu + 2n} = \sum_{r=0}^{\infty} \frac{(\mu + \nu + 2n + 2r)(\mu + \nu + 2n + r - 1)!}{r!} J_{\mu + \nu + 2n + 2r}$$

$$\therefore J_{\mu}(z)J_{\nu}(z)$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^n \Gamma(\mu + \nu + 2n + 1) \Gamma(\mu + \nu + 2n + 2r + 1) \Gamma(\mu + \nu + 2n + r)}{n! r! \Gamma(\mu + n + 1) \Gamma(\nu + n + 1) \Gamma(\mu + \nu + n + 1) \Gamma(\mu + \nu + 2n + 2r)}$$

$$J_{\mu + \nu + 2n + 2r}.$$

$$= \Omega_0 J_{\mu + \nu} + \Omega_{-2} J_{\mu + \nu + 2} + \Omega_{-4} J_{\mu + \nu + 4} + \dots + \Omega_{-2K} J_{\mu + \nu + 2K} + \dots \quad [A].$$

From [A] we can easily determine the coefficients $\Omega_0, \Omega_{-2}, \Omega_{-4}, \dots, \Omega_{-2k}$, &c., in terms of series of Gamma functions. In fact Ω_0 is obtained by putting $n=0, r=0$; Ω_{-2} by putting $n=1, r=0$; and $r=1, n=0$ and adding up the results. In general Ω_{-2k} is obtained by summing up the terms for $n=0, r=k$; $n=1, r=k-1$;; $n=k, r=0$. Thus

$$\Omega_0 = \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)},$$

$$\Omega_{-2} = \frac{\Gamma(\mu+\nu+3)}{\Gamma(\mu+\nu+2)} \left\{ \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+1)\Gamma(\nu+1)} + (-) \frac{\Gamma(\mu+\nu+3)}{\Gamma(\mu+2)\Gamma(\nu+2)} \right\},$$

&c., &c., &c.

$$\Omega_{-2k} = \frac{\Gamma(\mu+\nu+2k+1)}{\Gamma(\mu+\nu+2k)} \left\{ \frac{\Gamma(\mu+\nu+1)\Gamma(\mu+\nu+k)}{(0!k!\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\mu+\nu+1)} \right.$$

$$+ (-)^1 \frac{\Gamma(\mu+\nu+3)\Gamma(\mu+\nu+k+1)}{1!(k-1)!\Gamma(\mu+2)\Gamma(\nu+2)\Gamma(\mu+\nu+2)}$$

$$+ (-)^2 \frac{\Gamma(\mu+\nu+5)\Gamma(\mu+\nu+k+2)}{2!(k-2)!\Gamma(\mu+3)\Gamma(\nu+3)\Gamma(\mu+\nu+3)} + \&c.,$$

$$\left. + (-)^k \frac{\Gamma(\mu+\nu+2k+1)\Gamma(\mu+\nu+2k)}{k!0!\Gamma(\mu+k+1)\Gamma(\nu+k+1)\Gamma(\mu+\nu+k+1)} \right\},$$

$(k+1)$ terms in all.

From the general series for Ω_{-2k} we can write down any coefficient, that is to say, for Ω_n we take the first term putting $k=0$, for Ω we take the first two terms putting $k=1$, and so on. Hence the coefficients are determinable and we establish

$$J_\mu(z)J_\nu(z) = \sum_{r=0}^{\infty} \Omega_{-2r} J_{\mu+\nu+2r}(z)$$

Again

$$J_\lambda(z)J_\mu(z)J_\nu(z) = \sum_{r=0}^{\infty} \Omega_{-2r} J_\lambda(z)J_{\mu+\nu+2r}(z)$$

$$= \Omega_0 J_\lambda(z)J_{\mu+\nu}(z) + \Omega_{-2} J_\lambda(z)J_{\mu+\nu+2}(z) + \dots$$

$$+ \Omega_{-2k} J_\lambda(z)J_{\mu+\nu+2k}(z) + \&c$$

Suppose

$$J_{\lambda}(z) J_{\mu+\nu}(z) = \Omega'_0 J_{\lambda+\mu+\nu}(z) + \Omega'_{-2} J_{\lambda+\mu+\nu+2}(z) + \dots \\ + \Omega'_{-2k'} J_{\lambda+\mu+\nu+2k'}(z) + \&c. ;$$

$$J_{\lambda}(z) J_{\mu+\nu+2}(z) = \Omega'_{0,2} J_{\lambda+\mu+\nu+2}(z) + \Omega'_{-2,2} J_{\lambda+\mu+\nu+4}(z) + \dots \\ + \Omega'_{-2k',2} J_{\lambda+\mu+\nu+2k'+2}(z) + \&c. ;$$

$$J_{\lambda}(z) J_{\mu+\nu+4}(z) = \Omega'_{0,4} J_{\lambda+\mu+\nu+4}(z) + \Omega'_{-2,4} J_{\lambda+\mu+\nu+6}(z) + \dots \\ + \Omega'_{-2k',4} J_{\lambda+\mu+\nu+2k'+4}(z) + \&c. ; \\ \&c., \qquad \&c., \qquad \&c.$$

$$J_{\lambda}(z) J_{\mu+\nu+2k}(z) = \Omega'_{0,2k} J_{\lambda+\mu+\nu+2k}(z) + \Omega'_{-2,2k} J_{\lambda+\mu+\nu+2k+2}(z) \\ + \dots + \Omega'_{-2k',2k} J_{\lambda+\mu+\nu+2k+2k'}(z) + \&c.$$

Therefore on substitution

$$J_{\lambda}(z) J_{\mu}(z) J_{\nu}(z) = \Omega_0 \left[\Omega'_0 J_{\lambda+\mu+\nu}(z) + \Omega'_{-2} J_{\lambda+\mu+\nu+2}(z) + \dots \right. \\ \left. + \Omega'_{-2k'} J_{\lambda+\mu+\nu+2k'}(z) + \dots \right] \\ + \Omega_{-2} \left[\Omega'_{0,2} J_{\lambda+\mu+\nu+2}(z) + \Omega'_{-2,2} J_{\lambda+\mu+\nu+4}(z) + \dots \right. \\ \left. + \Omega'_{-2k',2} J_{\lambda+\mu+\nu+2k'+2}(z) + \dots \right] \\ + \Omega_{-4} \left[\Omega'_{0,4} J_{\lambda+\mu+\nu+4}(z) + \Omega'_{-2,4} J_{\lambda+\mu+\nu+6}(z) + \dots \right. \\ \left. + \Omega'_{-2k',4} J_{\lambda+\mu+\nu+2k'+4}(z) + \dots \right] \\ \&c., \qquad \&c., \qquad \&c. \\ + \Omega_{-2k} \left[\Omega'_{0,2k} J_{\lambda+\mu+\nu+2k}(z) + \Omega'_{-2,2k} J_{\lambda+\mu+\nu+2k+2}(z) + \dots \right. \\ \left. + \Omega'_{-2k',2k} J_{\lambda+\mu+\nu+2k'+2k}(z) + \dots \right] \\ + \&c. \\ = \Omega''_0 J_{\lambda+\mu+\nu} + \Omega''_{-2} J_{\lambda+\mu+\nu+2}(z) + \dots + \Omega''_{-2k''} J_{\lambda+\mu+\nu+2k'}(z) \\ + \&c.$$

where

$$\Omega''_0 = \Omega_0 \Omega'_0,$$

$$\Omega''_{-2} = \Omega_0 \Omega'_{-2} + \Omega_{-2} \Omega'_{0,2},$$

$$\Omega''_{-4} = \Omega_0 \Omega'_{-4} + \Omega_{-2} \Omega'_{-2,2} + \Omega_{-4} \Omega'_{0,4},$$

etc.,

etc.,

etc.

$$\Omega''_{-2k} = \Omega_0 \Omega'_{-2k} + \Omega_{-2} \Omega'_{-2(k-1),2} + \Omega_{-4} \Omega'_{-2(k-2),4} +$$

$$+ \Omega_{-2k} \Omega'_{0,2k},$$

and so on ; that is the new coefficients are expressible in terms of series of products of known coefficients. Proceeding in a similar manner we obtain for $(n+1)$ factors

$$J_{\lambda_1}(z) J_{\lambda_2}(z) J_{\lambda_3}(z) \dots J_{\lambda_{n+1}}(z) = \Omega_0^{(n)} J_{\lambda_1 + \lambda_2 + \dots + \lambda_{n+1}}(z)$$

$$+ \Omega_{-2}^{(n)} J_{\lambda_1 + \lambda_2 + \dots + \lambda_{n+1} + 2}(z)$$

+ &c., say, where the coefficients $\Omega_0^{(n)}$, $\Omega_{-2}^{(n)}$, ... are determinable from those of the product of n factors. The general formula is

$$\prod_{n+1} J_{\lambda}(z) = \sum_{k=0}^{\infty} \Omega_{-2k}^{(n)} J_{\sum + 2k},$$

where \sum stands for $\lambda_1 + \lambda_2 + \dots + \lambda_{n+1}$.

From definition

$$J_p(z) = \sum_{r=0}^{\infty} (-)^r \frac{z^{p+2r}}{2^{p+2r} r! \Gamma(p+r+1)}$$

whence

$$\int J_p(z) dz = \sum_{r=0}^{\infty} (-)^r \frac{z^{p+2r+1}}{2^{p+2r} r! \Gamma(p+r+1)} \frac{1}{p+2r+1} \quad (1)$$

$$= a_1 J_{p+1} + a_3 J_{p+3} + a_5 J_{p+5} + \&c., \text{ say.}$$

where a_1, a_3, a_5, \dots are undetermined coefficients. Substituting the values of $J_{p+1}, J_{p+3}, \&c.$ in the above we find

$$\int J_p(z) dz = a_1 \left\{ \sum_{r=0}^{\infty} (-)^r \frac{z^{p+1+2r}}{2^{p+1+2r} r! \Gamma(p+r+2)} \right\}$$

$$a_3 \left\{ \sum_{r=0}^{\infty} (-)^r \frac{z^{p+3+2r}}{2^{p+3+2r} r! \Gamma(p+r+4)} \right\} + \&c. \quad (2)$$

Comparing the coefficients of $z^{p+1}, z^{p+3}, \&c.$, we obtain after necessary simplifications.

$$a_1 = a_3 = a_5 = \dots = 2$$

Hence

$$\int J_p(z) dz = 2 \sum_{r=0}^{\infty} J_{p+2r+1}(z). \quad (3)$$

Again

$$\int J_{\mu}(z) J_{\nu}(z) dz = \sum_{r=0}^{\infty} \int \Omega_{-2r} J_{\mu+\nu+2r}(z) dz.$$

$$= \Omega_0 \int J_{\mu+\nu}(z) dz + \Omega_{-2} \int J_{\mu+\nu+2}(z) dz + \Omega_{-4} \int J_{\mu+\nu+4}(z) dz + \&c.$$

$$= \Omega_0 \cdot 2 \sum_{r=0}^{\infty} J_{\mu+\nu+2r+1}(z) + \Omega_{-2} \cdot 2 \sum_{r=0}^{\infty} J_{\mu+\nu+2r+3}(z)$$

$$+ \Omega_{-4} \cdot 2 \sum_{r=0}^{\infty} J_{\mu+\nu+2r+5}(z) + \&c.$$

$$= 2\Omega_0 \left[J_{\mu+\nu+1}(z) + J_{\mu+\nu+3}(z) + J_{\mu+\nu+5}(z) + \&c... \right]$$

$$+ 2\Omega_{-2} \left[J_{\mu+\nu+3}(z) + J_{\mu+\nu+5}(z) + J_{\mu+\nu+7}(z) + \&c... \right]$$

$$+ 2\Omega_{-4} \left[J_{\mu+\nu+5}(z) + J_{\mu+\nu+7}(z) + J_{\mu+\nu+9}(z) + \&c. \right]$$

$$+ \&c.$$

$$\begin{aligned}
& +2\Omega_{-2k} \left[J_{\mu+\nu+2k+1}(z) + J_{\mu+\nu+2k+3}(z) + \&c... \right] + \&c. \\
& = 2(\Omega)'_0 J_{\mu+\nu+1}(z) + 2(\Omega)'_{-2} J_{\mu+\nu+3}(z) + \dots \\
& \quad + 2(\Omega)'_{-2k} J_{\mu+\nu+2k+1}(z) + \&c. \\
& = 2 \sum_{k=0}^{\infty} (\Omega)'_{-2k} J_{\mu+\nu+2k+1}(z),
\end{aligned}$$

where

$$\begin{aligned}
(\Omega)'_0 &= \Omega_0, \\
(\Omega)'_{-2} &= \Omega_0 + \Omega_{-2}, \\
(\Omega)'_{-4} &= \Omega_0 + \Omega_{-2} + \Omega_{-4}, \\
&\dots \quad \dots \quad \dots \\
(\Omega)'_{-2k} &= \Omega_0 + \Omega_{-2} + \Omega_{-4} + \dots + \Omega_{-2k}, \&c.
\end{aligned}$$

Proceeding in a similar manner with the three-product integral, we obtain

$$\int J_{\lambda}(z) J_{\mu}(z) J_{\nu}(z) dz = 2 \sum_{k=0}^{\infty} (\Omega)''_{-2k} J_{\lambda+\mu+\nu+2k+1},$$

where

$$\begin{aligned}
(\Omega)''_0 &= \Omega''_0, \\
(\Omega)''_{-2} &= \Omega''_0 + \Omega''_{-2}, \\
(\Omega)''_{-4} &= \Omega''_0 + \Omega''_{-2} + \Omega''_{-4}, \\
&\dots \quad \dots \quad \dots \\
(\Omega)''_{-2k} &= \Omega''_0 + \Omega''_{-2} + \Omega''_{-4} + \dots + \Omega''_{-2k}, \&c.
\end{aligned}$$

Hence, in general for $(n+1)$ -product-integral, we establish

$$\int J_{\lambda_1}(z) J_{\lambda_2}(z) J_{\lambda_3}(z) \dots J_{\lambda_{n+1}}(z) dz = 2 \sum_{k=0}^{\infty} (\Omega)^{(n)}_{-2k} J_{\sum + 2k+1}(z),$$

where \sum has the usual significance.

LONGITUDINAL VIBRATIONS OF A HOLLOW CYLINDER.

By

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1. The longitudinal vibrations of a thin circular cylinder have been discussed at great length by Lord Rayleigh.¹ A second approximation (retaining terms up to the square of the radius of the cylinder), generally known as Pochhammer's solution, has also been obtained by C. Chree.² The frequency equation for a solid cylinder of any radius is given in Love's Elasticity. The object of this paper is to obtain the general frequency equation for a hollow solid bounded by two co-axial circular cylinders.

2. We take the axis of the cylinder as the axis of z and (r, θ, z) the cylindrical coordinates of any point. Denoting the displacements by u_r, u_θ, u_z , we may assume,³ as usual,

$$\begin{aligned} u &= U e^{i(a z + p t)} \\ u_\theta &= V e^{i(a z + p t)} \end{aligned} \quad \dots (1)$$

$$u_z = W e^{i(a z + p t)}$$

}

In the case of longitudinal vibrations, we may put $U=0$ and take V and W independent of θ . We then have

$$\left. \begin{aligned} \Delta &= \left(\frac{\partial U}{\partial r} + \frac{U}{r} + i a W \right) e^{i(a z + p t)} \\ \omega_r = \omega_z = 0, 2\omega &= \left(i a U - \frac{\partial W}{\partial r} \right) e^{i(a z + p t)} \end{aligned} \right\} \quad \dots (2)$$

where ω has been put for ω_θ .

¹ Theory of Sound, Vol. I, Chap. VII.

² Quarterly J. of Math., Vol. 21 (1886).

³ Love's Elasticity, Art. 201.

The equations of motion in terms of Δ and ω are

$$\left. \begin{aligned} \frac{\partial^2 \Delta}{\partial r^2} + \frac{1}{r} \frac{\partial \Delta}{\partial r} + h^2 \Delta &= 0 \\ \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} - \frac{\omega}{r^2} + k^2 \omega &= 0 \end{aligned} \right\} \quad \dots (3)$$

$$\text{where } h^2 = \frac{\rho^2 \rho}{\lambda + 2\mu} - a^2, \quad k^2 = \frac{\rho^2 \rho}{\mu} - a^2 \quad \dots (4)$$

The solutions of the equations (3) may be written

$$\left. \begin{aligned} \Delta &= \left\{ A' J_0(hr) + B' Y_0(hr) \right\} e^{i(rz + pt)} \\ \omega &= \left\{ C' J_1(kr) + D' Y_1(kr) \right\} e^{i(rz + pt)} \end{aligned} \right\} \quad \dots (5)$$

From (2) and (5), we have

$$\frac{\partial U}{\partial r} + \frac{U}{r} + iaW = A' J_0(hr) + B' Y_0(hr)$$

$$iaU - \frac{\partial W}{\partial r} = C' J_1(kr) + D' Y_1(kr)$$

These are satisfied by

$$\left. \begin{aligned} U &= A \frac{\partial}{\partial r} J_0(hr) + B \frac{\partial}{\partial r} Y_0(hr) + C a J_1(kr) + D a Y_1(kr) \\ W &= A i a J_0(hr) + B i a Y_0(hr) \\ &\quad + \frac{iC}{r} \frac{\partial}{\partial r} \left\{ r J_1(kr) \right\} + \frac{iD}{r} \frac{\partial}{\partial r} \left\{ r Y_1(kr) \right\} \end{aligned} \right\} \quad (6)$$

where

$$A = -\frac{A'}{h^2 + a^2}, \quad B = -\frac{B'}{h^2 + a^2}$$

$$C = \frac{2C'}{(k^2 + a^2)i}, \quad D = -\frac{2D'}{(k^2 + a^2)i}$$

The tractions across any surface $r=r$ are given by

$$\begin{aligned} \widehat{rr} &= \lambda \Delta + 2\mu \frac{\partial u_r}{\partial r} \\ &\propto \lambda [A' J_0(hr) + B' Y_0(hr)] \\ &+ 2\mu \frac{\partial}{\partial r} \left[A \frac{\partial}{\partial r} J_0(hr) + Ca J_1(hr) + B \frac{\partial}{\partial r} Y_0(hr) + Da Y_1(hr) \right] \end{aligned}$$

which, after simplification,

$$\begin{aligned} &= A \left\{ \lambda' J_0(hr) + \frac{\mu'}{r} J_1(hr) \right\} + B \left\{ \lambda' Y_0(hr) + \frac{\mu'}{r} Y_1(hr) \right\} \\ &+ \frac{Ca}{r} \left\{ hr J_0(hr) - J_1(hr) \right\} + \frac{Da}{r} \left\{ hr Y_0(hr) - Y_1(hr) \right\}, \end{aligned}$$

$$\widehat{r\theta} = 0,$$

$$\begin{aligned} \widehat{rz} &= \mu \left\{ 2\omega + 2 \frac{\partial u_z}{\partial r} \right\} \\ &\propto \mu \left\{ 2C' J_1(hr) + 2D' Y_1(hr) + 2 \frac{\partial W}{\partial r} \right\} \end{aligned}$$

which reduces to

$$\begin{aligned} &+ i\mu \{ A. 2ah J_1(hr) + B. 2ah Y_1(hr) + C(h^2 - a^2) J_1(hr) + D(h^2 - a^2) \\ &Y_1(hr) \}, \end{aligned}$$

in which we have put

$$\lambda' = \lambda(h^2 + a^2) - 2\mu h^2$$

$$\mu' = 2\mu h$$

3. The notations for Bessels functions used in this paper are those of Gray and Mathews and in the simplifications involved in the following processes, use will be made of the ordinary recurrence-formulae¹ for the functions $J_n(x)$, $Y_n(x)$, $J_n'(x)$, $Y_n'(x)$ and some others derived from them. Use will also be made of the two theorems: ¹

$$(i) \quad J_{n+1}(x)Y_n(x) - J_n(x)Y_{n+1}(x) = \frac{1}{x}$$

$$(ii) \quad J_n(x)Y_n'(x) - Y_n(x)J_n'(x) = \frac{1}{x}$$

Gray and Mathews, Bessel's Functions, pp. 13, 14, 16.

Nielsen, Theorie der cylinderfunktionen, p. 24.

CASE I.—BOTH BOUNDARIES FREE.

4. If the boundaries $r=a$ and $r=b$ are both free from tractions, we have the following conditions:—

$$\begin{aligned}
 & A \left[\lambda' J_0(ha) + \frac{\mu'}{a} J_1(ha) \right] + B \left[\lambda' Y_0(ha) + \frac{\mu'}{a} Y_1(ha) \right] \\
 & + \frac{Ca}{a} \left[ka J_0(ka) - J_1(ka) \right] + \frac{Da}{a} \left[ka Y_0(ka) - Y_1(ka) \right] = 0 \\
 & A \left[\lambda' J_0(hb) + \frac{\mu'}{b} J_1(hb) \right] + B \left[\lambda' Y_0(hb) + \frac{\mu'}{b} Y_1(hb) \right] \\
 & + \frac{Cb}{b} \left[kb J_0(kb) - J_1(kb) \right] + \frac{Db}{b} \left[kb Y_0(kb) - Y_1(kb) \right] = 0 \\
 & A 2ah J_1(ha) + B 2ah Y_1(ha) + C(k^2 - a^2) J_1(ka) + D(k^2 - a^2) Y_1(ka) = 0 \\
 & A 2ah J_1(hb) + B 2ah Y_1(hb) + C(k^2 - a^2) J_1(kb) + D(k^2 - a^2) Y_1(kb) = 0
 \end{aligned}$$

Eliminating the constants A, B, C, D, we get the frequency equation—

$$\begin{array}{ll}
 \lambda' J_0(ha) + \frac{\mu'}{a} J_1(ha) & \lambda' J_0(hb) + \frac{\mu'}{b} J_1(hb) \\
 \lambda' Y_0(ha) + \frac{\mu'}{a} Y_1(ha) & \lambda' Y_0(hb) + \frac{\mu'}{b} Y_1(hb) \\
 \frac{a}{a} \left\{ ka J_0(ka) - J_1(ka) \right\} & \frac{a}{b} \left\{ kb J_0(kb) - J_1(kb) \right\} \\
 \frac{a}{a} \left\{ ka Y_0(ka) - Y_1(ka) \right\} & \frac{a}{a} \left\{ kb Y_0(kb) - Y_1(kb) \right\} \\
 2ah J_1(ha) & 2ah J_1(hb) \\
 2ah Y_1(ha) & 2ah Y_1(hb) \\
 (k^2 - a^2) J_1(ka) & (k^2 - a^2) J_1(kb) \\
 (k^2 - a^2) Y_1(ha) & (k^2 - a^2) Y_1(kb)
 \end{array} = 0 \quad (7)$$

or,

$$\left\{ \lambda' J_0(ha) + \frac{\mu'}{a} J_1(ha) \right\} \text{I} - \left\{ \lambda' J_0(hb) + \frac{\mu'}{b} J_1(hb) \right\} \text{II}$$

$$+ 2ahJ_1(ha) \text{III} - 2ahJ_1(hb) \text{IV} = 0$$

where

$$\text{I} = \left\{ \lambda' Y_0(hb) + \frac{\mu'}{b} Y_1(hb) \right\} (k^2 - a^2)^2 F_{a_1 b_1} + \frac{2a^2 h}{b} (k^2 - a^2) Y_1(ha)$$

$$+ \frac{2a^2 h}{b} (k^2 - a^2) Y_1(hb) \left\{ -kbF_{a_1 b_0} + F_{a_1 b_1} \right\}$$

$$\text{II} = \left\{ \lambda' Y_0(ha) + \frac{\mu'}{a} Y_1(ha) \right\} (k^2 - a^2)^2 F_{a_1 b_1}$$

$$- 2ahY_1(ha) \frac{a}{a} (k^2 - a^2) \left\{ kaF_{a_0 b_1} - F_{a_1 b_1} \right\} - 2ahY_1(hb) \frac{a}{a} (k^2 - a^2)$$

$$\text{III} = - \left\{ \lambda' Y_0(ha) + \frac{\mu'}{a} Y_1(ha) \right\} \frac{a}{b} (k^2 - a^2)$$

$$- \left\{ \lambda' Y_0(hb) + \frac{\mu'}{b} Y_1(hb) \right\} \frac{a}{a} (k^2 - a^2) \left\{ kaF_{a_0 b_1} - F_{a_1 b_1} \right\}$$

$$+ 2ahY_1(hb) \frac{a^2}{ab} \left\{ k^2 abF_{a_0 b_0} - kbF_{a_1 b_0} - kaF_{a_0 b_1} + F_{a_1 b_1} \right\}$$

$$\text{IV} = \left\{ \lambda' Y_0(ha) + \frac{\mu'}{a} Y_1(ha) \right\} \frac{a(k^2 - a^2)}{b} \left\{ -kbF_{a_1 b_0} + F_{a_1 b_1} \right\}$$

$$+ \left\{ \lambda' Y_0(hb) + \frac{\mu'}{b} Y_1(hb) \right\} \frac{a(k^2 - a^2)}{b}$$

$$+ 2ahY_1(ha) \frac{a^2}{ab} \left\{ k^2 abF_{a_0 b_0} - kbF_{a_1 b_0} - kaF_{a_0 b_1} + F_{a_1 b_1} \right\}$$

where

$$F_{a_r b_s} = F_{b_s a_r} = J_r(ka) Y_s(kb) - J_s(kb) Y_r(ka)$$

The frequency-equation may be finally put in the form¹

$$\begin{aligned}
 & \left(k^2 - \alpha^2 \right)^2 F_{a_1 b_1} \left[\lambda'^2 G_{a_0 b_0} + \lambda' \mu' \left\{ \frac{1}{a} G_{a_0 b_1} + \frac{1}{b} G_{a_1 b_0} \right\} \right. \\
 & \qquad \qquad \qquad \left. + \frac{\mu'^2}{ab} G_{a_1 b_1} \right] \\
 & + \frac{2\alpha^2 h}{b} \left(k^2 - \alpha^2 \right) \left\{ -kb F_{a_1 b_0} + F_{a_1 b_1} \right\} \left\{ \lambda' G_{a_0 b_1} + \frac{\mu'}{a} G_{a_1 b_1} \right\} \\
 & + \frac{2\alpha^2 h}{a} \left(k^2 - \alpha^2 \right) \left\{ ka F_{a_0 b_1} - F_{a_1 b_1} \right\} \left\{ \lambda' G_{b_0 a_1} - \frac{\mu'}{b} G_{a_1 b_1} \right\} \\
 & + \frac{4\alpha^4 h^2}{ab} \left\{ k^2 ab F_{a_0 b_0} - kb F_{a_1 b_0} - ka F_{a_0 b_1} + F_{a_1 b_1} \right\} G_{a_1 b_1} \\
 & - \frac{4\alpha^2 (k^2 - \alpha^2)}{ab} = 0 \dots \quad (8)
 \end{aligned}$$

where

$$G_{a_r b_s} \equiv -G_{b_s a_r} \equiv J_r(ha) Y_s(hb) - J_s(hb) Y_r(ha)$$

CASE II.—ONE BOUNDARY RIGID.

5. If the cylinder be free at the surface $r=a$ and rigidly clamped at $r=b$, the boundary conditions are

$$\begin{aligned}
 & A \left[\lambda' J_0(ha) + \frac{\mu'}{a} J_1(ha) \right] + B \left[\lambda' Y_0(ha) + \frac{\mu'}{a} Y_1(ha) \right] \\
 & + \frac{Ca}{a} \left[ka J_0(ka) - J_1(ka) \right] + \frac{Da}{a} \left[ka Y_0(ka) - Y_1(ka) \right] = 0 \\
 & A 2ah J_1(ha) + B 2ah Y_1(ha) + C(k^2 - \alpha^2) J_1(ha) + D(k^2 - \alpha^2) Y_1(ha) = 0 \\
 & A h J_1(hb) + B h Y_1(hb) - Ca J_1(kb) - Da Y_1(kb) = 0 \\
 & A a J_0(hb) + B a Y_0(hb) + C h J_0(kb) + D h Y_0(kb) = 0
 \end{aligned}$$

¹ Here, as also in other cases, the actual process of simplification being rather long, the intermediate steps have been omitted.

Eliminating A, B, C, D, we have the frequency-equation

$$\lambda' J_0(ha) + \frac{\mu}{a} J_1(ha) \quad , \quad 2ahJ_1(ha) \quad ,$$

$$\lambda' Y_0(ha) + \frac{\mu'}{a} Y_1(ha) \quad 2ahY_1(ha)$$

$$\frac{a}{a} \left\{ kaJ_0(ha) - J_1(ha) \right\} \quad (k^2 - a^2)J_1(ha)$$

$$\frac{a}{a} \left\{ kaY_0(ha) - Y_1(ha) \right\} , \quad (k^2 - a^2)Y_1(ha) \quad ,$$

$$hJ_1(hb) \quad aJ_0(hb) \quad ;=0$$

$$hY_1(hb) \quad aY_0(hb)$$

$$-aJ_1(hb) \quad , \quad hJ_0(hb)$$

$$-aY_1(hb) \quad , \quad hY_0(hb)$$

After simplification, this equation will reduce to

$$\begin{aligned} \frac{2a^2\lambda'}{b} - \frac{a^2(k^2 - a^2)}{ab} - kh(k^2 - a^2)F_{a_1b_0} \left\{ \lambda'G_{a_0b_1} + \frac{\mu'}{a}G_{a_1b_1} \right\} \\ - a^2(k^2 - a^2)F_{a_1b_1} \left\{ \lambda'G_{a_0b_1} + \frac{\mu'}{a}G_{a_1b_0} \right\} + \frac{2a^2h^2k}{a}G_{a_1b_1} \\ \left\{ kaF_{a_1b_0} - F_{a_1b_0} \right\} \\ + \frac{2a^4h}{a}G_{a_1b_0} \left\{ kaF_{a_0b_1} - F_{a_1b_1} \right\} = 0 \quad \dots \quad (9) \end{aligned}$$

6. The equations (8) and (9) do not admit of exact solutions. Approximate solutions *by trial* may be obtained for assumed values of the ratio $a:b$, by making use of the tables for the values of $J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$. The actual work of calculation will of course be very

complicated. The tables of $J_0(x)$ and $J_1(x)$ are given by Meissel¹ and those of $Y_0(x)$ and $Y_1(x)$ by Airey.² This method has been adopted by Mr. Southwell³ in the numerical calculation of some of the approximate values of the period in the case of the transverse vibrations of an annular disc, where, in addition to the ordinary Bessel and Neumann functions, the corresponding functions with imaginary arguments also appear in the frequency equation.

CASE III.—THICKNESS OF THE SHELL VERY SMALL.

7. When the thickness of the shell is very small, we may write $a+da$ for b , expand the functions containing $(a+da)$ in ascending powers of da , and, to a first approximation, neglect all powers of da beyond the first.

Performing these operations in the equation (7), we obtain the frequency-equation for a thin shell of radius a in the form

$$\lambda' J_0(ha) + \frac{\mu'}{a} J_1(ha) \qquad -\lambda' h J_1'(ha) - \frac{\mu'}{a^2} J_2(ha)$$

$$\lambda' Y_0(ha) + \frac{\mu'}{a} Y_1(ha) \qquad -\lambda' k Y_1(ha) - \frac{\mu'}{a^2} Y_2(ha)$$

$$ak J_1'(ka)$$

$$ak^2 J_1''(ka)$$

$$ak Y_1'(ka)$$

$$ak^2 Y_1''(ka)$$

$$2ah J_1(ha) \qquad 2ah^2 J_1'(ha) \qquad | = 0$$

$$2ah Y_1(ha) \qquad 2ah^2 Y_1'(ha)$$

$$(k^2 - a^2) J_1(ka) \qquad (k^2 - a^2) J_1'(ka)$$

$$(k^2 - a^2) Y_1(ka) \qquad (ka^2 - a^2) Y_1'(ka)$$

¹ Reproduced in Gray and Mathew's Bessel's Functions.

² Rep. Brit. Assoc. (1914).

³ Proc. Roy. Soc., Ser. A, Vol. 101 (1922), p. 133.

which gives

$$\begin{aligned}
 & \left\{ \frac{(k^2 - a^2)^2 \lambda'^2 h}{ka} \left\{ J_1(ha) Y_0(ha) - J_0'(ha) Y_1(ka) \right\} \right. \\
 & + \frac{\lambda' \mu' (k^2 - a^2)^2}{ka^3} \left\{ J_2(ha) Y_0'(ha) - J_0'(ha) Y_2(ha) \right\} \\
 & + \frac{\mu'^2 (k^2 - a^2)^2}{ka^3} \left\{ J_2(ha) Y_1(ha) - J_1(ha) Y_2(ha) \right\} \\
 & + 2ha^2 k \lambda' (k^2 - a^2) H \left\{ J_0(ha) Y_1(ha) - J_1(ha) Y_0(ha) \right\} \\
 & + \frac{2\lambda' a^2 h^2 (k^2 - a^2)}{a^2} \left\{ J_0(ha) Y_1'(ha) - J_1'(ha) Y_0(ha) \right\} \\
 & + \frac{2\mu' a^2 h^2 (k^2 - a^2)}{a^3} \left\{ J_1(ha) Y_1'(ha) - Y_1(ha) J_1'(ha) \right\} \\
 & + \frac{2\lambda' a^2 h^2 (k^2 - a^2)}{ka} \left\{ -J_1(ha) Y_1'(ha) + J_1'(ha) Y_1(ha) \right\} \\
 & + \frac{2\mu' a^2 h^2 (k^2 - a^2)}{a^3} \left\{ J_1'(ha) Y_2(ha) - J_2(ha) Y_1'(ha) \right\} \\
 & + 4a^4 h^2 k^2 H \left\{ J_1(ha) Y_1'(ha) - Y_1(ha) J_1'(ha) \right\} = 0
 \end{aligned}$$

where

$$\begin{aligned}
 H &= J_1'(ka) Y_1''(ka) - J_1''(ka) Y_1'(ka) \\
 &= \frac{1}{4} \left[\left\{ J_1'(ka) - \frac{1}{ka} J_1(ka) \right\} \left\{ -3Y_1(ka) + Y_3(ka) \right\} \right. \\
 &\quad \left. - \left\{ Y_0(ka) - \frac{1}{ka} Y_1(ka) \right\} \left\{ -3J_1(ka) + J_3(ka) \right\} \right] \\
 &= \frac{1}{4} \left[3 \left\{ J_1(ka) Y_0'(ka) - J_0'(ka) Y_1(ka) \right\} + \left\{ J_0(ka) Y_3(ka) \right. \right. \\
 &\quad \left. \left. - Y_0(ka) J_3(ka) \right\} + \frac{1}{ka} \left\{ J_3(ka) Y_1(ka) - J_1(ka) Y_3(ka) \right\} \right] \\
 &= \frac{1}{ka} - \frac{1}{k^3 a^3}
 \end{aligned}$$

This can be further simplified into

$$\begin{aligned}
 & - \frac{(k^2 - a^2)^2 \lambda'^2}{k a^3} + \frac{2 \lambda' \mu' (k^2 - a^2)^2}{k h^2 a^3} + \frac{\mu'^2 (k^2 - a^2)^2}{h k a^3} \\
 & \quad - \frac{2 a^2 k^2 \lambda' (k^2 - a^2)}{a} \left(\frac{1}{k a} - \frac{1}{k^3 a^3} \right) \\
 & + \frac{2 \lambda' a^2}{a^4} + \frac{2 \mu' a^2 h (k^2 - a^2)}{a^3} - \frac{2 \lambda' a^2 h^2 (k^2 - a^2)}{a^2} \\
 & \quad + \frac{4 a^4 h^2 k^2}{a} \left(\frac{1}{k a} - \frac{1}{k^3 a^3} \right) \\
 & - \frac{2 \mu' a^2 (k^2 - a^2)}{a^5} = 0
 \end{aligned}$$

or, multiplying throughout by a^5 , we have

$$\begin{aligned}
 & a^3 \left\{ 4 a^4 h^2 k^2 - 2 \lambda' a^2 h^2 (k^2 - a^2) - 2 \lambda' k a^2 (k^2 - a^2) - \frac{1}{k} \lambda'^2 (k^2 - a^2)^2 \right\} \\
 & + a^2 \left\{ \frac{2}{h h^2} \lambda' \mu' (k^2 - a^2)^2 + 2 h \mu^2 a^2 (k^2 - a^2) \right\} \\
 & + a \left\{ \frac{1}{h k} \mu'^2 (k^2 - a^2) + \frac{2}{k} \lambda' a^2 (k^2 - a^2) + 2 \lambda' a^2 - 4 a^4 h^2 \right\} \\
 & + 2 \mu' a^2 (k^2 - a^2) = 0
 \end{aligned}$$

If the tube is of very small bore, and we may neglect all powers of a beyond the first, the frequency equation is

$$a \left\{ \frac{1}{h k} \mu'^2 (k^2 - a^2) + \frac{2}{k} \lambda' a^2 (k^2 - a^2) + 2 \lambda' a^2 - 4 a^4 h^2 \right\} + 2 \mu' a^2 (k^2 - a^2) = 0.$$

My thanks are due to Prof. S. N. Basu, at whose suggestion I took up the work.

**A GENERAL THEOREM FOR THE REPRESENTATION OF X ,
WHERE X REPRESENTS THE POLYNOMIAL $\frac{x^p-1}{x-1}$.**

By

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(1) It is a well known theorem that if p is an odd prime and if X represents the polynomial $\frac{x^p-1}{x-1}$, there is a remarkable transformation of X , which may be expressed as the identity,

$$4X = Y^2 - (-1)^{\frac{p-1}{2}} pZ^2,$$

where Y and Z are polynomials in x with integral co-efficients. This identity is known as Gauss's Identity and much has been written about it by different distinguished mathematicians including Gauss, Legendre, L. J. Rogers, G. B. Mathews and others.

There is another identity

$$27X = f(U, V, W)$$

where U , V and W are rational integral polynomials in x . This identity has been given by Mr. Eisentein. Without knowing Mr. Eisentein's result¹ I discovered the same identity in a different form.

The object of this paper is to show a general method with the help of which many other formulae of similar type can be easily discovered. The well known Gauss's identity and the cubic identity are only particular cases of this general theorem. It is believed that many other general formulae of transformation will be obtained later on.

This method can be usefully applied only to those values of q for which the cyclotomic section has been completely solved,

¹ I did not know that the same problem had been worked out by Mr. Eisenstein; it has just been pointed out to me by a referee of London Mathematical Society.

Throughout this paper I have adhered to the notation of Mr. W. Burnside as given in the Proceedings of the London Mathematical Society, 1915.

(2) Let $X_1, X_2, X_3, \dots, X_q$ be the period values of cyclotomic q -section, [I mean that X_1, X_2, \dots, X_q have the same meaning as given by G. B. Mathews in proving the Gauss's identity], and let it be supposed that $\eta_0, \eta_1, \eta_2, \dots, \eta_{q-1}$ are the roots of the period equation.

(3) I have taken the following notations from the paper of Mr. W. Burnside :

p is an odd prime

q is an odd prime factor, and $p-1=qt$.

ω is an assigned primitive p'^h root of unity.

α is an assigned primitive root of the congruence $\alpha^{p-1} \equiv 1 \pmod{p}$.

β is an assigned primitive root of the congruence $\beta^{q-1} \equiv 1 \pmod{q}$.

Each of the $p-1$ primitive p'^h roots of unity is included just once in the form $\omega^{\alpha^i + \beta^j}$, $i=0, 1, \dots, q-1$; $j=0, 1, \dots, t-1$.

$$\omega^{\alpha^i + \beta^j} \quad (i=0, 1, \dots, q-1; \quad j=0, 1, \dots, t-1).$$

Put

$$A_i = \sum_{j=0}^{t-1} \omega^{\alpha^i + \beta^j} \quad (i=0, 1, \dots, q-1).$$

Each A_i consists of the sum of t distinct primitive p'^h roots of unity, and each primitive p'^h root occurs just once in one of the A_i 's. When ω is replaced by ω^{α^2} , each A_i remains unaltered. When ω is replaced by ω^{α} , A_i 's undergo the cyclical permutation

$$(A_0 A_1 \dots A_{q-1})$$

If ω' is any root occurring in A_i , then

$$A_i = \sum_{j=0}^{t-1} \omega'^{\alpha^i + \beta^j}.$$

In particular since t is even, if A_i contains ω' it will also contain ω'^{-1} .

When i is replaced by β , the A'_s undergo the permutation

$$\begin{pmatrix} A_0 & A_1 & A_2 \dots A_{q-1} \\ A_0 & A_\beta & A_{2\beta} \dots A_{(q-1)\beta} \end{pmatrix}$$

where the suffixes are reduced (mod. q). This leaves A unchanged and gives a regular circular permutation of the other A'_s .

If A_i and A_j are two distinct A 's and if the product of A_i and A_j is formed without reduction, *i.e.*, without taking account of the relation

$$1 + \omega + \omega^2 + \dots + \omega^{p-1} = 0,$$

it will consist of the product of t^2 primitive p^{t-1} roots because ω occurs in A_i , then ω^{t-1} does not occur in A_j . Moreover since $A_i A_j$ is unaltered when ω is replaced by ω^{t^2} , the product can be arranged as the sum of a number of A 's.

Hence

$$A_i A_j = \sum_{k=1}^{k=t} C_{i,j,k} A_k$$

where C 's are zeroes or positive integers, such that

$$\sum_{k=1}^{k=t} C_{i,j,k} = t.$$

The product

$$A_i^2 = t + \sum_{k=1}^{k=t} C_{i,i,k} A_k$$

where again the C 's are zeroes or positive integers, and

$$\sum_{k=1}^{k=t} C_{i,i,k} = t-1$$

(4) In particular, the square, cube, etc., of A 's can always be represented as the sum of A 's. It follows therefore that it is always possible to represent the square, cube, etc., of $\eta_0, \eta_1, \eta_2, \dots$ and η_{q-1} as the sum of a number of $\eta_0, \eta_1, \eta_2, \dots$ and η_{q-1} . Thus we can form q equations which can be always solved uniquely because they are linear simultaneous equations in

$$\eta_0, \eta_1, \eta_2, \dots \text{ and } \eta_{q-1}$$

Therefore X_1, X_2, \dots , and X_q can always be expressed in the form

$$U + V\eta + W\eta^2 + \dots + M\eta^{q-1}.$$

where U, V, W, \dots, M are polynomials in x with integral coefficients.

What has just been established shows at once that the following is always a possible operation :

$$X_1 = U + V\eta_0 + W\eta_0^2 + \dots + M\eta_0^{q-1}.$$

$$X_2 = U + V\eta_1 + W\eta_1^2 + \dots + M\eta_1^{q-1}.$$

$$X_3 = U + V\eta_2 + W\eta_2^2 + \dots + M\eta_2^{q-1}.$$

$$X_q = U + V\eta_{q-1} + \dots + M\eta_{q-1}^{q-1}.$$

Now it is well known that

$$\begin{aligned} X &= X_1 X_2 X_3 \dots X_q \\ &= (U + V\eta_0 + W\eta_0^2 + \dots + M\eta_0^{q-1}) \times \\ &\quad \left(U + V\eta_1 + W\eta_1^2 + \dots + M\eta_1^{q-1} \right) \times \dots \\ &\quad \dots \times \left(U + V\eta_{q-1} + W\eta_{q-1}^2 + \dots + M\eta_{q-1}^{q-1} \right). \\ &= U^q + V^q \eta_0 \eta_1 \eta_2 \dots \eta_{q-1} + W^q \eta_0^2 \eta_1^2 \eta_2^2 \dots \eta_{q-1}^2 + \dots \\ &\quad \dots + M^q \eta_0^{q-1} \eta_1^{q-1} \eta_2^{q-1} \dots \eta_{q-1}^{q-1} \dots \quad (A) \end{aligned}$$

the symmetric functions involved in equation (A) can always be determined by the method given in any standard book on theory of equations.

Calculating the symmetric functions in the equation (A) and substituting them in it we find the required formula. Now this general formula will be applied in two particular cases in order to illustrate the use of this method.

When $q=2$, we get

$$\begin{aligned} X &= X_1 X_2 = (U + V\eta_0) (U + V\eta_1) \\ &= U^2 + V^2 \eta_0 \eta_1 + UV(\eta_0 + \eta_1). \end{aligned}$$

Substituting the values of η_0 , η_1 and $\eta_0 + \eta_1$ from the theory of cyclotomic bi-section we get the well known Gauss's Identity.

In order to prove the theorem when $q=3$ let us put for X_1 , X_2 and X_3 their values corresponding to cyclotomic periods. Let us suppose that η_0 , η_1 and η_2 are the roots of the period equation

$$\eta^3 + \eta^2 - \frac{p-1}{3}\eta - \frac{1}{9}\left(pa' + \frac{p-1}{3}\right) = 0$$

where p is a prime number of the form $6n+1$. Then X_1 is a polynomial of which the coefficients are symmetric functions of the roots of $X=0$, the sum of which makes up $\eta_0=0$. Similar statement holds good for X_2 and X_3 .

Let us suppose for a moment that η_0 , η_1 and η_2 are subject to the same conditions which are true for η_0 and η_1 in finding the transformation formula

$$tX = Y^2 - (-1)^{\frac{p-1}{2}} pZ^2$$

Then it is evident that the coefficients of X_1 may all be reduced to the form $a + b\eta_0$. Similarly the coefficients of X_2 and X_3 can also be represented.

Therefore we have identically

$$X_1 = U + V\eta_0.$$

$$X_2 = U + V\eta_1.$$

$$X_3 = U + V\eta_2.$$

where U and V are polynomials in x with integral coefficients and η_0 , η_1 and η_2 are the roots of the period equation

$$\eta^3 + \eta^2 - \frac{p-1}{3}\eta - \frac{1}{9}\left(pa' + \frac{p-1}{3}\right) = 0$$

$$\begin{aligned} \therefore X = X_1 X_2 X_3 &= (U + V\eta_0)(U + V\eta_1)(U + V\eta_2) \\ &= U^3 + \sum \eta_0 U^2 V + \sum \eta_0 \eta_1 UV^2 + \eta_0 \eta_1 \eta_2 V^3 \\ &= (3U - V)^3 - pV^2(9U - 3a'V - V) \quad \dots \quad (1) \end{aligned}$$

Let us now suppose that the condition to which the investigation given above is subject, has been removed and let 3 be a factor of $p-1$: because p is a prime of the form $6n+1$ this operation is always possible. Let it be supposed that $t = \epsilon^{\frac{p-1}{3}}$, ω a primitive p th root of unity and a

a primitive root of the congruence $a^{p-1} \equiv 1 \pmod{p}$. Then each of the $p-1$ primitive roots of unity is involved only once in the form

$$\frac{a^{i+3^x}}{\omega} \quad (i=0, 1, 2; \quad x=0, 1, \dots, t)$$

$$\text{Put } \Delta_i = \sum_{x=0}^{p-t-1} \frac{a^{i+3^x}}{\omega} \quad (i=0, 1, 2)$$

Then each Δ_i consists of the sum of t distinct primitive p th. root of unity and each primitive p th. root occurs only once. It is very well known that the product of Δ 's can always be represented as the sum of Δ 's and hence in particular the square of Δ 's. Hence it is always possible to represent η_0^2 as the sum of η_0 , η_1 and η_2 .

$$\therefore \eta_0^2 = m + a\eta_0 + b\eta_1 + c\eta_2 \quad \dots \quad (B)$$

where m , a , b , and c are integers and some of them may be zeroes.

From the theory of cyclotomic tri-section it is evident that the roots η_0 , η_1 and η_2 are connected by the following linear relation:—

$$\eta_0 + \eta_1 + \eta_2 = -1 \quad \dots \quad (C)$$

By the help of the equations (B) and (C) it is always possible to represent η_1 and η_2 in terms of η_0^2 , η_0 and some integers.

$\therefore X_1$ can be represented as $U + V\eta_0 + W\eta_0^2$ where U , V and W are polynomials in x with integral coefficients.

Similarly

$$X_2 = U + V\eta_1 + W\eta_1^2$$

$$\text{and} \quad X_3 = U + V\eta_2 + W\eta_2^2$$

$$\therefore X = X_1 X_2 X_3$$

$$\begin{aligned} &= (U + V\eta_0 + W\eta_0^2)(U + V\eta_1 + W\eta_1^2) + (U + V\eta_2 + W\eta_2^2) \\ &= U^3 + U^2V(\eta_0 + \eta_1 + \eta_2) + U^2W(\eta_0^2 + \eta_1^2 + \eta_2^2) \\ &\quad + UV^2(\eta_0\eta_1 + \eta_0\eta_2 + \eta_1\eta_2) + UW^2(\eta_0^2\eta_1^2 + \eta_0^2\eta_2^2 \\ &\quad \quad \quad + \eta_1^2\eta_2^2) \\ &\quad + UVW(\eta_1\eta_0^2 + \eta_0\eta_1^2 + \eta_0^2\eta_2 + \eta_1^2\eta_2 + \eta_0\eta_2^2 + \eta_1\eta_2^2) \\ &\quad + V^3\eta_0\eta_1\eta_2 + V^2W(\eta_0^2\eta_1\eta_2 + \eta_0\eta_1^2\eta_2 + \eta_0\eta_1\eta_2^2) \\ &\quad + VW^2(\eta_0\eta_1^2\eta_2^2 + \eta_0^2\eta_1^2\eta_2 + \eta_0^2\eta_1\eta_2^2) + W^3\eta_0^2\eta_1^2\eta_2^2 \end{aligned}$$

Now calculating the symmetric coefficients $\Sigma \eta_0$, $\Sigma \eta_0^2$, $\Sigma \eta_0 \eta_1$, etc. and substituting the values in the equation just obtained and simplifying it we get

$$\begin{aligned}
 27X = & 27U^3 - 27U^2V + (18p+9)U^2W - (9p-9)UV^2 \\
 & - 9\left(p\alpha' - \frac{2p-2}{3}\right)UVW + 3\left\{(p-1)^2 + 2p\alpha' + \frac{2p-2}{3}\right\}UW^2 \\
 & + 3\left(p\alpha' + \frac{p-1}{3}\right)V^3 \\
 & - 3\left(p\alpha' + \frac{p-1}{3}\right)V^2W - (p-1)\left(p\alpha' + \frac{p-1}{3}\right)VW^2 \\
 & + \frac{1}{3}\left(p\alpha' + \frac{p-1}{3}\right)^2W^3
 \end{aligned}$$

If in this equation W becomes zero, then this formula reduces to the formula (1) obtained above. It is evident that the value of U can not be equal to zero for any prime and hence we can not obtain any formula by supposing U to be equal to zero. It is also evident that whenever W is not zero, V also can not be equal to zero.

Now to establish this theorem in the case when $q=4$ let us put for X_1, X_2, X_3 and X_4 their values corresponding to cyclotomic periods. Let us suppose that η_0, η_1, η_2 and η_3 are the roots of the period equation of cyclotomic quartic-section. Then X_1 is a polynomial of which the coefficients are symmetric functions of the roots of $X=0$, the sum of which makes up $\eta_0=0$. Similar statements hold for X_2, X_3 and X_4 .

It is possible, as in the previous case, to represent η_0^2 as the sum of η_0, η_1, η_2 and η_3

$$\therefore \eta_0^2 = m + a\eta_0 + b\eta_1 + c\eta_2 + d\eta_3 \quad \dots \quad (D)$$

$$\text{and} \quad \eta_0^3 = m' + a'\eta_0 + b'\eta_1 + c'\eta_2 + d'\eta_3 \quad \dots \quad (E)$$

where $m, m', a, a', b, b', c, c'$ and d, d' are integers, some of which may be zeroes.

From the theory of cyclotomic quartic-section it is evident that the roots η_0, η_1, η_2 and η_3 are connected by the following linear relation:—

$$\eta_0 + \eta_1 + \eta_2 + \eta_3 = -1 \quad \dots \quad (F)$$

By the help of the equations (D), (E) and (F) it is always possible to represent η_1 and η_2 in terms of $\eta_0^3, \eta_0^2, \eta_0$ and some integers

$\therefore X_1$ can be represented as $U + V\eta_0 + W\eta_0^2 + Y\eta_0^3$, where U, V, W and Y are polynomials in x with integral coefficients.

Similarly

$$X_2 = U + V\eta_1 + W\eta_1^2 + Y\eta_1^3,$$

$$X_3 = U + V\eta_2 + W\eta_2^2 + Y\eta_2^3,$$

and

$$X_4 = U + V\eta_3 + W\eta_3^2 + Y\eta_3^3.$$

$$\therefore X = X_1 X_2 X_3 X_4$$

$$= (U + V\eta_0 + W\eta_0^2 + Y\eta_0^3)(U + V\eta_1 + W\eta_1^2 + Y\eta_1^3).$$

$$(U + V\eta_2 + W\eta_2^2 + Y\eta_2^3)(U + V\eta_3 + W\eta_3^2 + Y\eta_3^3)$$

$$= U^4 + U^3V\sum\eta_0 + U^3W\sum\eta_0^2 + U^3Y\sum\eta_0^3 + U^2V^2\sum\eta_0\eta_1 +$$

$$U^2W^2\sum\eta_0^2\eta_1^2 + U^2Y^2\sum\eta_0^3\eta_1^3 + U^2VW\sum\eta_0\eta_1^2 +$$

$$U^2VY\sum\eta_0\eta_1^3 +$$

$$U^2WY\sum\eta_0^2\eta_1^3 + UV^3\sum\eta_0\eta_1\eta_2 + UV^2W\sum\eta_0\eta_1\eta_2^2 +$$

$$UV^3Y\sum\eta_0\eta_1\eta_2^3 +$$

$$UVW^2\sum\eta_0\eta_1^2\eta_2^2 + UVY^2\sum\eta_0\eta_1^3\eta_2^3 +$$

$$UVWY\sum\eta_0\eta_1^2\eta_2^3 + UW^3\sum\eta_0^2\eta_1^2\eta_2^2 +$$

$$UW^2Y\sum\eta_0^2\eta_1^2\eta_2^3 + UWY^2\sum\eta_0^3\eta_1^3\eta_2^2 +$$

$$UY^3\sum\eta_0^3\eta_1^3\eta_2^3 + V^4\eta_0\eta_1\eta_2\eta_3 + V^3W\sum\eta_0\eta_1\eta_2\eta_3^2 +$$

$$V^3Y\sum\eta_0\eta_1\eta_2\eta_3^3 + V^2W^2\sum\eta_0\eta_1\eta_2^2\eta_3^2 +$$

$$V^2Y^2\sum\eta_0\eta_1\eta_2^3\eta_3^3 + V^2WY\sum\eta_0\eta_1\eta_2^2\eta_3^3 +$$

$$VW^3\sum\eta_0\eta_1^2\eta_2^2\eta_3^3 + VW^2Y\sum\eta_0\eta_1^3\eta_2^2\eta_3^3 +$$

$$VWY^2\sum\eta_0\eta_1^2\eta_2^3\eta_3^3 +$$

$$VY^3\sum\eta_0\eta_1^3\eta_2^3\eta_3^3 + W^4\eta_0^2\eta_1^2\eta_2^2\eta_3^2 + W^3Y$$

$$\sum\eta_0^3\eta_1^3\eta_2^3\eta_3^3 +$$

$$W^2Y^2\sum\eta_0^2\eta_1^2\eta_2^3\eta_3^3 + WY^3\sum\eta_0^3\eta_1^3\eta_2^3\eta_3^3 +$$

$$Y^4\eta_0^3\eta_1^3\eta_2^3\eta_3^3.$$

Now calculating the symmetric coefficients $\Sigma \eta_0$, $\Sigma \eta_0 \eta_1$, etc., and substituting the values in the equation just obtained and by simplifying it we get

$$\begin{aligned}
 X = & U^4 - U^3V + (1-2q)U^3W + (3q-3r-1)U^3Y + \\
 & qU^2V^2 + (q^2-2r+2s)U^2W^2 + \\
 & (3s+q^3+3r^2-3qr-3qs)U^2Y^2 + \\
 & (3r-q)U^2VW + (q-2q^2-r+4s)U^2VY + \\
 & (2r-q^2+qr-5s)U^2WY - rUV^3 + \\
 & (r^2-2qs)UW^3 + (3qrs-r^3-3s^2)UY^3 + \\
 & (4qs+qr-3s-3r^2)UVWY + \\
 & (r-4s)UV^2W + (2qr-r+s)UV^2Y + \\
 & (3s-qr)UW^2V + (2r^2+qs-q^2r-5sr)UVY^2 + \\
 & (2qs+sr-r^2)UW^2Y + \\
 & (qr^3-2q^2s-sr+4s^2)UWY^3 + sV^4 - sV^3W + \\
 & (s-2qs)V^3Y + qsV^3W^2 + s(q^2-2r+2s)V^3Y^2 + \\
 & (3sr-qs)V^3WY - srVW^3 + (sr-4s^2)VW^2Y + \\
 & (3s^2-qrs)VWY^3 + (sr^2-2qs^2)VY^3 + s^3W^4 \\
 & - s^3W^3Y + s^3qW^2Y^2 - s^2rWY^3 + s^3Y^4
 \end{aligned}$$

or

$$256X = f(U, V, W, Y)$$

It should be noted here that the period equation of cyclotomic quarti-section is supposed to be

$$\eta^4 + \eta^3 + q\eta^2 + r\eta + s = 0;$$

and all the symmetric functions involved in the quartic identity given above have been expressed in the terms of the coefficients of the period equation. The coefficients of the period equation, however, can always be determined by the formulae given by A. Cayley, V. S. Le. Resque, Charlotte Angas Scott, W. Burnside.

Putting $q=5, 6, 7$, etc., we can obtain as many identities as we like but the calculation of symmetric functions involved becomes unmanageable.

I have calculated the values of U, V, and W for the primes 13 and 31 in the cubic identity given above. Similarly the values U, V and W for other primes can also be calculated.

Calculation for the prime 13.

It is well known that

$$\eta_0 = \omega + \omega^3 + \omega^{12} + \omega^6,$$

$$\eta_1 = \omega^5 + \omega^9 + \omega^7 + \omega^4.$$

and

$$\eta_2 = \omega^{10} + \omega^2 + \omega^8 + \omega^{11}.$$

We may take any of them. Let us take the first.

Then

$$(x-\omega)(x-\omega^3)(x-\omega^{12})(x-\omega^6)=0$$

$$\therefore x^4 - \eta_0 x^3 + (\eta_1 + 2)x^2 - \eta_0 x + 1 = 0$$

$$\eta_0 + \eta_1 + \eta_2 = -1 \quad \dots (1)$$

$$\eta_0^2 = \eta_2 + 2\eta_1 + 4 \quad \dots (2)$$

Solving the equations (1) and (2), we obtain

$$\eta_1 = \eta_0^2 + \eta_0 - 3.$$

and

$$\eta_2 = -\eta_0^2 - 2\eta_0 + 2.$$

$$\therefore x^4 - \eta_0 x^3 + (\eta_0^2 + \eta_0 - 1)x^2 - \eta_0 x + 1 = 0.$$

In this case we do not require the value of η_2 .

$$\therefore U = x^4 - x^3 + 1.$$

$$V = -x^3 + x^2 - x.$$

and

$$W = x^2.$$

If we substitute these values in the cubic identity we find that the identity is satisfied.

As this is an identity we may put $x=1$.

and then

$$U=1,$$

$$V=-1,$$

and

$$W=1.$$

Substituting these values in the identity we get

$$27 \times 13 = 27\{1+1+0-4-7+14+1+1-4+1\}.$$

CALCULATION FOR THE PRIME 31.

$$\eta_0 = \omega + \omega^{27} + \omega^{16} + \omega^{29} + \omega^8 + \omega^{30} + \omega^4 + \omega^{15} + \omega^2 + \omega^{23},$$

$$\eta_1 = \omega^3 + \omega^{19} + \omega^{17} + \omega^{25} + \omega^{24} + \omega^{28} + \omega^{13} + \omega^{14} + \omega^6 + \omega^7,$$

and $\eta_2 = \omega^9 + \omega^{26} + \omega^{20} + \omega^{12} + \omega^{10} + \omega^{22} + \omega^5 + \omega^{11} + \omega^{18} + \omega^{21}.$

Then $(x-\omega)(x-\omega^{27})(x-\omega^{16})(x-\omega^{29})(x-\omega^8)(x-\omega^{30})$

$$\times (x-\omega^4)(x-\omega^{15})(x-\omega^2)(x-\omega^{23})=0.$$

$$\therefore x^{10} - \eta_0 x^9 + (\eta_0 + 2\eta_1 + \eta_2 + 5)x^8 - (5\eta_0 + 3\eta_1 + 4\eta_2)x^7$$

$$+ (10 + 5\eta_0 + 8\eta_1 + 7\eta_2)x^6 - (9\eta_0 + 7\eta_1 + 9\eta_2 + 2)x^5$$

$$+ (10 + 5\eta_0 + 8\eta_1 + 7\eta_2)x^4 - (5\eta_0 + 3\eta_1 + 4\eta_2)x^3$$

$$+ (5 + \eta_0 + 2\eta_1 + \eta_2)x^2 - \eta_0 x + 1 = 0.$$

$$\eta_0 + \eta_1 + \eta_2 = -1 \quad \dots (1)$$

$$\eta_0^2 = 10 + 4\eta_1 + 3\eta_0 + 2\eta_2 \quad \dots (2)$$

Solving the equations (1) & (2) we obtain

$$\eta_1 = \frac{\eta_0^2 - \eta_0 - 8}{2},$$

and

$$\eta_2 = \frac{6 - \eta_0^2 - \eta_0}{2}.$$

Substituting these values we get

$$2x^{10} - 2\eta_0 x^9 + (\eta_0^2 - \eta_0)x^8 + (\eta_0^2 - 3\eta_0)x^7 + (\eta_0^2 - 5\eta_0 - 2)x^6$$

$$+ (2\eta_0^2 - 2\eta_0 - 2)x^5 + (\eta_0^2 - 5\eta_0 - 2)x^4 + (\eta_0^2 - 3\eta_0)x^3$$

$$+ (\eta_0^2 - \eta_0)x^2 - 2\eta_0 x + 2 = 0.$$

$$\therefore U = 2x^{10} - 2x^9 - 2x^5 - 2x^4 + 2,$$

$$V = -2x^9 - x^8 - 3x^7 - 5x^6 - 2x^5 - 5x^4 - 3x^3 - x^2 - 2x,$$

and

$$W = x^8 + x^7 + x^6 + 2x^5 + x^4 + x^3 + x^2.$$

Now putting $x=1$,

$$U = -2,$$

$$V = -24,$$

and

$$W = 8,$$

Substituting these values in the cubic identity we get

$$27 \times 31 = 27 \times \frac{1}{8} \{-8 + 96 + 672 + 11520 - 5376 - 14848 \\ - 110592 - 36864 + 32768 + 122880\}.$$

Here also we see that the identity is satisfied.

QUARTIC IDENTITY.

I have calculated the values of U, V, W and Y for the primes 13 and 17, which for other primes also can be calculated in a similar way.

For the prime 13.

The period equation of cyclotomic quarti-section for the prime 13 is

$$\eta^4 + \eta^3 + 2\eta^2 - 4\eta + 3 = 0.$$

The value of quarti-sectional period equation for each prime under 100 has been given by A. Cayley in the Proceedings of London Mathematical Society. And for other primes they can be calculated by the formula given by Miss Scott in the American Journal of Mathematics, VIII.

The formula is as follows :—

$$\eta^4 + \eta^3 - \left\{ \frac{1}{2}(p-1) + l + m \right\} \eta^2 + \frac{1}{2} \{ p(l-m) - (l+m) \} \eta \\ - \frac{1}{4} \{ p(l-m)^2 - (l+m)^2 \} = 0.$$

But in the quartic identity given above we have supposed the period equation to be

$$\eta^4 + \eta^3 + q\eta^2 + r\eta + s = 0.$$

Hence

$$q = - \left\{ \frac{1}{2}(p-1) + l + m \right\},$$

$$r = \frac{1}{2} \{ p(l-m) - (l+m) \},$$

and

$$s = - \frac{1}{4} \{ p(l-m)^2 - (l+m)^2 \}.$$

It is well known that

$$\eta_0 = \omega + \omega^3 + \omega^9,$$

$$\eta_1 = \omega^3 + \omega^9 + \omega^5,$$

$$\eta_2 = \omega^4 + \omega^{12} + \omega^{10},$$

and

$$\eta_3 = \omega^5 + \omega^{11} + \omega^7.$$

Then

$$(x-\omega)(x-\omega^3)(x-\omega^9)=0.$$

$$\therefore x^3 - \eta_0 x^2 + \eta_2 x - 1 = 0$$

$$\eta_0 + \eta_1 + \eta_2 + \eta_3 = -1 \quad \dots (1)$$

$$\eta_1 + 2\eta_2 - \eta_0^3 = 0 \quad \dots (2)$$

$$\eta_0 + 3\eta_1 + 3\eta_2 + 6 - \eta_0^3 = 0 \quad \dots (3)$$

Solving the equations (1), (2) and (3) we obtain

$$\eta_2 = \frac{3-2\eta_0-\eta_0^3}{3}.$$

Here in this case we do not require the values of η_1 and η_3 .

$$\therefore x^3 - \eta_0 x^2 + \eta_2 x - 1 = 0 \text{ becomes}$$

$$3x^3 - 3\eta_0 x^2 + (3-2\eta_0-\eta_0^3)x - 3 = 0$$

$$\therefore U = 3x^3 + 3x - 3,$$

$$V = -(3x^2 + 2x),$$

$$W = 0,$$

and

$$Y = -x.$$

If we substitute these values in the quartic identity we find that it is satisfied.

Now if we put $x=1$,

$$U=3,$$

$$V=-5,$$

$$W=0,$$

and

$$Y=-1.$$

Substituting these values in the identity we get

$$\begin{aligned} 13 = \frac{1}{81} \{ & 81 + 135 + 0 - 459 + 450 + 0 + 639 + 0 + 450 + 0 - 1500 + 0 \\ & + 105 + 0 + 0 + 675 + 0 - 1710 + 0 + 0 + 1875 + 0 - 1125 \\ & + 0 + 1350 + 0 + 0 + 0 + 0 + 60 + 0 + 0 + 0 + 0 + 27 \} \end{aligned}$$

Calculation for the prime 17.

The period equation of cyclotomic quarti-section for the prime 17 is

$$\eta^4 + \eta^3 - 6\eta^2 - \eta + 1 = 0.$$

And

$$\eta_0 = \omega + \omega^{13} + \omega^{16} + \omega^4,$$

$$\eta_1 = \omega^3 + \omega^5 + \omega^{14} + \omega^{12},$$

$$\eta_2 = \omega^9 + \omega^{15} + \omega^8 + \omega^2,$$

and

$$\eta_3 = \omega^{10} + \omega^{11} + \omega^7 + \omega^6.$$

Then

$$(x-\omega)(x-\omega^{1^2})(x-\omega^{1^3})(x-\omega^4)=0.$$

$$\therefore x^4 - \eta_0 x^3 + (2 + \eta_1)x^2 - \eta_0 x + 1 = 0.$$

$$\eta_0 + \eta_1 + \eta_2 + \eta_3 = -1 \quad \dots (1)$$

$$\eta_0^2 = \eta_2 + 2\eta_1 + 4 \quad \dots (2)$$

$$\eta_0^3 = 9\eta_0 + \eta_1 + 3\eta_2 + 3\eta_3 \quad \dots (3)$$

Solving the equations (1), (2) and (3) we obtain

$$\eta_1 = \frac{1}{2}(6\eta_0 - 3 - \eta_0^3).$$

Here we do not require the value of η_2 and η_3 .

$$x^4 - \eta_0 x^3 + (2 + \eta_1)x^2 - \eta_0 x + 1 = 0 \text{ becomes}$$

$$x^4 - \eta_0 x^3 + \frac{1}{2}(6\eta_0 + 1 - \eta_0^3)x^2 - \eta_0 x + 1 = 0$$

$$\text{or } 2x^4 - 2\eta_0 x^3 + (6\eta_0 + 1 - \eta_0^3)x^2 - 2\eta_0 x + 2 = 0.$$

$$U = 2x^4 + x^2 + 2.$$

$$V = -2x^3 + 6x^2 - 2x,$$

$$W = 0,$$

$$Y = -x^2.$$

If we substitute these values in the quartic identity we find that it is satisfied.

Now if we put $x=1$,

$$U=5,$$

$$V=2,$$

$$W=0,$$

and

$$Y=-1.$$

Substituting these values in the identity we get

$$\begin{aligned} 17 = & \frac{1}{16} \{ 625 - 250 + 0 + 2000 - 600 + 0 - 5250 + 0 + 3650 + 0 + 40 \\ & + 0 - 80 + 0 + 0 - 280 + 0 + 370 + 0 + 0 + 16 + 0 - 104 \\ & + 0 + 160 + 0 + 0 + 0 + 0 - 26 + 0 + 0 + 0 + 0 + 1 \}. \end{aligned}$$

The quartic identity may also be looked upon as a general formula in quartic forms, because with its help any number of primes of the form $4n+1$, where n is a positive integer, can be represented in a quartic form as has been shown above.

"ON THE COVARIANT CURVES OF A SINGULAR n -ic"

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§ 1

If $f=0$ represents an algebraic curve of the n^{th} degree, its *Hessian*, *Steinerian* and *Cayleyan* are represented by covariants of the form f and are therefore called *covariant curves*. There may, of course, be other curves which are also represented by covariants of the original quantic;¹ but in view of the simplicity of the relations which these curves bear to the original and the detail in which they have been studied, we shall throughout be dealing with them alone whenever we speak of covariant curves. Many beautiful properties of these curves have been established² but the central problem in regard to them is however the determination of their Plucker's numbers. This problem has been completely solved and the Plucker's numbers of the covariant curves tabulated in most of the standard treatises on the subject,³ for the case where the original curve is non-singular. But when the given curve is not non-singular no attempt seems to have been made to calculate systematically all the characteristics of the three curves nor, is this surprising, in view of the fact that in addition to the order of each curve we have to determine directly two characteristics of each curve

¹ For example Zeuthen has considered the curve enveloped by the tangents of the first polars of y (point of Steinerian) at the double points. See

CLEBSCH AND LINDEMANN : *Leçons sur la Géométrie*.

Fr. Trans., t. 2 ; Ch. I ; IV.

² Vide Clebsch-Lindemann ; Ibid Ch. I ; Section IV also SALMON, *Higher Plane Curves* . pp 363-8.

³ See *Encyclopadie der Mathematischen Wiss.* § III C 4. No. 7. pp 339-42.

or the deficiency of any one¹ of them and *one* other characteristic for each directly. Only some scattered results seem to have been obtained in this direction.² We have attempted this problem here and believe that we have been able to solve it successfully.

Coming back to the case of the non-singular primitive curve, it has been pointed out above that the problem of the determination of the Pluckerian characteristics has been completely solved. In fact there are two distinct methods by means of which this has been achieved. We shall briefly indicate them here.

1°. SALMON'S METHOD: Assume that the *Hessian* has in general no double points.³ The order of the *Hessian* being known *a priori* this assumption enables us to calculate all its characteristics as well as its deficiency and this in virtue of Riemann's theorem, already referred to, will enable the calculation of the characteristics of the other two curves knowing their order and class.

2°. CLEBSCH'S METHOD: This method is indirect and has for its basis the determination of one characteristic of the *Steinerian* in addition to its class and order, *viz.*, the number of its inflexional tangents. In fact, what is done is to take any point and find out the tact-invariant of its first polar and the *Hessian*; this tact-invariant equated to zero would give the point equation of the *Steinerian*. The degree of this tact-invariant in terms of the coefficients of the first polar would then be given by $3(n-2)(5n-11)$ and observing that the tact-invariant also includes the inflexional tangents of the *Steinerian*, the number of these last is given by

$$3(n-2)(5n-11) - 3(n-2)^2 = 3(n-2)(4n-9).$$

The next step is to calculate the deficiency of the *Steinerian*, and then pass on to the other two curves with the aid of Riemann's theorem.

We here indicate another entirely different method of obtaining these results and this consists in proving directly that the *Cayleyan* has no inflexional tangents. This enables us to calculate all the

¹ In virtue of Riemann's famous theorem relative to the invariance of the deficiency of a curve in all undeterminative transformations, the three covariant curves have the same deficiency.

See "*Clebsch—Lindemann*": Ibid; t. 3.; Ch. I (1)

² See *Koehler: Bull Soc Math de France* i (1873), pp. 124-9 for the determination of the class of the *Steinerian* when the given curve has multiple and higher singularities

See also HILTON: *Plane Algebraic Curves*; p. 106.

³ As regards this assumption see the remarks by Clebsch: *Ibid* t. 2.

characteristics of this curve as well as its deficiency. We then pass on to the *Steinerian* and *Hessian* and also calculate their Plucker's numbers.

§ 2

We first proceed with the case where the original curve is non-singular.

Let C, C_1, C_2, C_3 denote respectively the original curve, its *Hessian*, *Steinerian* and *Cayleyan*. Confining to Clebsch's notation which we shall use throughout, let

n =order of C

k =class „

d =no. of double points of C

t =no. of double tangents of C .

r = „ „ cusps of C .

w = „ „ inflexional tangents of C .

ρ =deficiency or genus of C

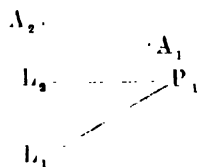


Fig. 1.

and let the same letters with the subscripts 1, 2 or 3 denote the corresponding characteristics of the *Hessian*, *Steinerian* or *Cayleyan* respectively.

We shall now proceed to prove that $w_3=0$. In fact, if c_3 had an inflexional tangent, then, at the corresponding inflexion-point two of its tangents would be coincident. It requires therefore that corresponding to this inflexional tangent two corresponding points of C_1 and C_2 would coincide or the corresponding points would be double points. It follows therefore that to a double point of C_2 there corresponds a double point of C_1 . If therefore we can show that double points of C_2 take birth from the fact that to two separate points of C_1 , corresponds a single point of C_2 and inversely that to a double point on C_2 correspond two separate points on C_1 , we can conclude that to a double point of C_2 cannot correspond a double point of C_1 and hence C_3 has no inflexional tangents.

Nor is it difficult to prove the assumption we have made above. We shall show that the pole of every first polar with two double points must be a node of C_3 . Suppose that a first polar curve has two double points A_1 and A_2 (Fig. 1) which must necessarily lie on C_1 . Remembering that the polar line of a double point of the first polar touches C_2 at a point of which it is the first polar, we deduce that the polar line of A_1 is a line L_1 touching C_2 at a point P_1 . Similarly the polar

line of A_2 is a line L_2 touching C_2 at the same point P_1 , for, otherwise, one and the same curve will have to be the first polar of two distinct points. Hence the double points A_1 and A_2 give a node on C_2 with L_1 and L_2 as nodal tangents; L_1 and L_2 are necessarily distinct lines for, A_1 and A_2 must be a pair of poles of a line touching C_2 at two points.

Conversely the first polars of any two points, except P_1 , on L_1 touch at A_1 and similarly those of any two points on L_2 must touch at another point A_2 . Also, the first polar of P_1 , considered as a point on L_1 , has a double point at A_1 while the same first polar when P_1 is considered a point on L_2 has a double point at A_2 . Hence the first polar of a node P_1 on C_2 has two double points A_1 and A_2 lying on C_1 . Hence to a double point on C_2 correspond two distinct points on C_1 and *vice versa*; and this is the theorem which we set out to establish.

We can write therefore $w_3=0$

$$\text{but } n_3=3(n-2)(5n-11)$$

$$\text{and } k_3=3(n-1)(n-2).$$

Knowing three of the characteristics of C_3 it is now easy to deduce all the others. We have, in fact,

$$d_3=\frac{9}{2}(n-2)(5n-13)(5n^2-19n+16)$$

$$r_3=18(n-2)(2n-5)$$

$$t_3=\frac{9}{2}(n-2)^2(n^2-2n-1)$$

and we also have

$$p_3=\frac{1}{2}(3n-7)(3n-8)$$

Proceeding next to the curve C_3 we have, in virtue of Riemann's theorem already referred to;

$$p_3=p_3=\frac{1}{2}(3n-7)(3n-8)$$

and *a priori* $n_2=3(n-2)^2$

$$k_3=3(n-1)(n-2)$$

and these three equations enable us to determine all the other characteristics of C_2 . We have actually, with the aid of Plücker's equations :

$$d_2 = \frac{3}{2}(n-2)(n-3)(3n^2-9n-5)$$

$$r_2 = 12(n-2)(n-3)$$

$$t_2 = \frac{3}{2}(n-2)(n-3)(3n^2-3n-8)$$

and $w_2 = 3(n-2)(4n-9)$

Similarly for the curve C_1 we have

$$p_1 = p_2 = p_3 = \frac{1}{2}(3n-7)(3n-8)$$

and also $n_1 = 3(n-2)$

so that it only remains to determine one other characteristic of C_1 ; but we have

$$p_1 = \frac{1}{2}(n_1-1)(n_1-2) - d_1 - r_1$$

and putting $n_1 = 3(n-2)$

$$\frac{1}{2}(n_1-1)(n_1-2) = \frac{1}{2}(3n-7)(3n-8)$$

so that we deduce $d_1 + r_1 = 0$

$$\text{i.e.}, d_1 = 0 : r_1 = 0$$

It is now easy to determine all the other characteristics of C_1 . We have, in fact,

$$k_1 = 3(n-2)(3n-7)$$

$$t_1 = \frac{27}{2}(n-1)(n-2)(n-3)(3n-8)$$

$$w_1 = 9(n-2)(3n-8)$$

These results may now be tabulated (See Table I).

TABLE 1.

	Hessian.	Steinerian.	Cayleyan.
n	$3(n-2)$	$3(n-2)^2$	$3(n-2)(5n-11)$
k	$3(n-2)(n-7)$	$3(n-1)(n-2)$	$3(n-1)(n-2)$
d	0	$\frac{3}{2}(n-2)(n-3)(3n^2-9n-5)$	$\frac{9}{2}(n-2)(5n-3)(5n^2-19n+16)$
l	$\frac{27}{2}(n-1)(n-2)(n-3)(3n-8)$	$\frac{3}{2}(n-2)(n-3)(3n^2-5n-8)$	$\frac{9}{2}(n-2)^2(n^2-2n-1)$
r	0	$12(n-2)(n-3)$	$18(n-2)(2n-5)$
w	$9(n-2)(3n-8)$	$3(n-2)(4n-9)$	0

§ 3

Let us now consider the case where the original curve is not non-singular but has only d double points and r cusps, there being no other higher singularities.

We have seen indirectly in §2 that when the original curve is non-singular the curve C_1 has in general (*i.e.*, if we exclude special relations between the coefficients of the original curve) no double points. A direct proof of this does not as yet seem to have been established.¹ It may, however, happen that a particular relation among the coefficients of the primitive curve may be different from that obtained by expressing the condition for a double point so that C_1 may have a double point without the primitive curve possessing any. Thus, for example, considering the n -ic

$$c_3^n + b_3 c_3^{n-1} + c_2^2 c_3^{n-2} + d_2^3 c_3^{n-3} + \dots = 0$$

(b_2, c_2^2, d_2^3 being binary quantics in x_1, x_2) it is easy to show that the first polar of a point $P(1,0,0)$ will have a node at $Q(0,0,1)$ if $b_0=0$; $c_0=c_1=0$ and further that the curve does not pass through P . Thus the curve c_1 should pass through Q and it can be shown further that Q is a node on c_1 provided $d_0=d_1=0$. Thus it appears that the *Hessian* of a curve has a double point at $(0,0,1)$ without that point lying on the curve at all.²

As stated above, this is due to the fact that the particular relation $d_0=d_1=0$ is different from the condition necessary for the possessing of double points.

We shall, however, assume that the *Hessian* has, in general, no double points when the original curve is non-singular.

Now, a double point on the original curve transforms into a double point on C_1 and a cusp into a triple point. Moreover, since the triple point has two distinct branches only and the other touching one of these branches, the triple point is equivalent to two nodes and one cusp. Thus,

d nodes on C are d nodes on C_1

r cusps $2r$ nodes and

r cusps on C_1 .

¹See "*Clebsch-Lindemann*:" t. 2, Section IV.

²For another such example see "*Clebsch-Lindemann*." *Ibid.*

Therefore, when the original curve is not non-singular, we can write

$$d_1 = d + 2r$$

Further
$$u_1 = 3(n-2)$$

so that we can proceed with the determination of the characteristics of C_1 completely.

We have, in fact,

$$k_1 = 3(n-2)(n-7) - 2d - 7r$$

$$w_1 = 9(n-2)(3n-8) - 6d - 20r$$

$$t_1 = \frac{27}{2}(n-1)(n-2)(n-3)(3n-8)$$

$$-2d(3n^2 - 27n + 46) - 3r(7n^2 - 63n + 101) + (2d + 7r)^2.$$

We can also calculate the deficiency of C_1 as

$$p_1 = \frac{1}{2}(3n-7)(3n-8) - d - 3r$$

Proceeding next to the curve C_2 , we have, in virtue of Riemann's Theorem

$$\rho_2 = p_1 = \frac{1}{2}(3n-7)(3n-8) - d - 3r$$

Further,

$$u_2 = 3(n-2)^2$$

so that it only remains to determine one; other Plucker's number of C_2 when the primitive is not non-singular. But it is a well-known result¹ that the class of the *Steinerian* is in this case equal to

$$k_2 = 3(n-1)(n-2) - 2d - 4r$$

¹ See *Hilton : Ibid.* p. 106.

The other Plucker's characteristics now follow easily. We can write

$$d_2 = \frac{3}{2} (n-2)(n-3)(3n^2-9n-5) + d + 3r$$

$$r_2 = 12(n-2)(n-3) - 2r$$

$$w_2 = 3(n-2)(4n-9) - 6d - 14r$$

$$t_2 = \frac{3}{2} (n-2)(n-3)(3n^2-3n-8)$$

$$-2d(3n^2-9n+7) - r(12n^2-36n+29) + 2(d+2r)^2.$$

It now remains to calculate the Plucker's numbers of the *Cayleyan*. The available data are

$$n_3 = 3(n-2)(5n-11)$$

$$p_3 = p_2 = p_1 = \frac{1}{2} (3n-7)(3n-8) - d - 3r$$

so that, as in the case of C_2 , it is sufficient to know one other characteristic of C_3 by direct methods.

A method immediately suggests itself by considering the arguments by means of which we deduced in §2 that $w_3=0$, when the primitive is non-singular. In fact, that proof depended on showing that to a double point on c_2 does not also correspond a double point on C_1 but a pair of distinct points on C_1 . Now even when the original curve is not non-singular this argument need not in any way be modified and we can therefore write in this case too

$$w_3 = 0$$

and the other characteristics of C_3 are now easily determined. We find on actual calculation that

$$d_3 = \frac{9}{2} (n-2)(5n-13)(5n^2-19n+16) - 2d - 6r$$

$$r_3 = 18(n-2)(2n-5) - 3d - 9r$$

$$t_3 = \frac{9}{2} (n-2)^2 (n^2-2n-1) - \delta\phi_1(n) - k\phi_2(n) \\ + (A\delta+Bk)^2$$

where ϕ_1 , ϕ_2 , A and B are obtained by an easy simplification.

Herewith is appended a tabulated list of these Plucker's numbers (See Table II),

TABLE II.

	Hessian.	Steinerian.	Cayleyan.
u	$3(u-2)$	$3(u-2)^2$	$3(u-2)(5u-11)$
v	$3(u-2)(v-7)-2d-7r$	$3(u-1)(u-2)-2d-4r$	$3(u-1)(u-2)+d+3r$
d	$d+2r$	$\frac{3}{2}(u-2)(u-3)(3u^2-9u-5)+d+3r$	$\frac{9}{2}(u-2)(5u+3)(5u^2-19u+16)-2d-6r$
t	See the paper proper.		
r		$12(v-2)(u-3)-2r$	$18(u-2)(2u-5)-3d-9r$
w	$9(u-2)(3u-8)-6d-20r$	$3(v-2)(4u-9)-6d-14r$	0

EQUITENSE TRANSFORMATIONS ABOUT A FIXED POINT TAKEN AS ORIGIN.

BY

C. E. CULLEN.

[Summary. Equitense transformations in ordinary 3-way space Ω_4 (which include *reflexions*, *rotations* and *translations*) are first defined, and are divided into *rigid transformations* and *pseudo-rigid transformations*, — a rigid transformation being an equitense transformation which can be regarded as a resultant of infinitesimal equitense transformations. Those which take place about a fixed finite origin O (divided into *rotations* and *pseudo-rotations* about O) are then discussed in greater detail. In connection with the complete interpretations of rotations and pseudo-rotations about O whose equations are known, special attention may be directed to the theorems of Arts. 4 and 5, in which those interpretations are given in forms characterised by perfect symmetry and freedom from ambiguity. The paper concludes by explaining how pseudo-rigid transformations in Ω_4 can be regarded as rigid transformations in Ω_3 , — a reflexion about a plane in Ω_4 being equivalent to a rotation about that plane through two right angles in Ω_3 .]

1. Equitense transformations ; rigid and pseudo-rigid transformations.

We take O to be a fixed origin accessible to an observer situated in ordinary 3-way space $\Omega = \Omega_4$ (of rank 4) ; and (OX, OY, OZ) to be a *right-handed* set of rectangular axes drawn from O in Ω . All points and all transformations will be supposed to be *real* ; and it will be left to the reader to gather when these restrictions are unnecessary.

We may regard (OX, OY, OZ) as a mathematical abstraction derived from a man standing upright with outstretched arms and looking forwards, O being the base of the head, OX being the right arm, OY being drawn horizontally in the direction of vision, and OZ being drawn vertically upwards through the head. The rotation about OZ which carries OX to OY through a positive right angle is right-handed, and OZ is the right-handed axis of that rotation. If OX' is the left arm, then (OX', OY, OZ) are a set of left-handed rectangular axes ; the rotation about OZ which carries OX' to OY through a positive right angle is left-handed ; and OZ is the left-handed axis of that rotation. To make the results of this paper applicable when $(OX,$

OY, OZ) are a left-handed set of rectangular axes, the terms 'right-handed' and 'left-handed' must be interchanged whenever they occur.

A projective transformation in Ω is a transformation which converts the points of Ω into the points of an ordinary 3-way space Ω' coincident with Ω in such a way as to establish a one-one correspondence between:

all finite points of Ω and all finite points of Ω'
all infinite points of Ω and all infinite points of Ω' :

every finite point of Ω being converted into the corresponding finite point of Ω' , and every point in the plane at infinity of Ω being converted into the corresponding point in the plane at infinity of Ω' . If the point P whose co-ordinates with reference to the axes OX, OY, OZ are x, y, z is converted into the point P₁ whose co-ordinates with reference to the same axes are x_1, y_1, z_1 , the general equation of such a transformation (as applied to finite points) can be expressed in the form

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \text{ (equivalent to } \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = M^{-1} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} \text{)}, \quad \dots \quad (1)$$

where

$$M = \begin{bmatrix} l_1 & l_2 & l_3 & p \\ m_1 & m_2 & m_3 & q \\ n_1 & n_2 & n_3 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \dots \quad (2)$$

is an undegenerate square matrix whose elements are all finite.

An equitense transformation in Ω is a projective transformation in Ω which leaves the (positive or undirected) distance between every two finite points of Ω unchanged. It necessarily leaves the angles between any two straight lines or any two planes of Ω unchanged. If we put

$$\phi = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}, \quad \phi' = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \dots \quad (3)$$

the transformation (1) will be equitense if and only if the equation

$$(x'_1 - x_1)^2 + (y'_1 - y_1)^2 + (z'_1 - z_1)^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2$$

is an identity in x, y, z, x', y', z' when

$$\begin{bmatrix} x_1' - x_1 \\ y_1' - y_1 \\ z_1' - z_1 \end{bmatrix} = \phi \cdot \begin{bmatrix} x' - x \\ y' - y \\ z' - z \end{bmatrix},$$

i.e. if and only if

$$\phi' \phi = I = \phi \phi', \quad \text{or} \quad \phi' = \phi^{-1} \quad \dots (4)$$

Accordingly (1) will be the general equation of an equitense transformation (supposed to be real) when and only when ϕ is a real square semi-unit matrix. We then have

$$\det \phi = \pm 1;$$

and the equitense transformation will be called

a *rigid transformation* when $\det \phi = 1$

a *pseudo-rigid transformation* when $\det \phi = -1$.

It can be applied only to those points of Ω which form a configuration S_1 . It then converts the points of S into the points of another configuration S_1 lying in Ω .

Clearly all equitense transformations constitute a group G , which is a sub-group of the group of projective transformations; and all rigid transformations constitute a group H , which is a sub-group of G .

The equitense transformation (1) will be called :

(1) the *identical transformation* (which leaves all points of Ω unchanged) when M is the unit matrix of order 4;

(2) an *infinitesimal transformation* (which gives only an infinitesimal displacement to every point of Ω) when we can put

$$M = \begin{bmatrix} 1 + \lambda_1 & l_2 & l_3 & p \\ m_1 & 1 + \mu_2 & m_3 & q \\ n_1 & n_2 & 1 + \nu_3 & r \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where all the letters denote infinitesimal scalar numbers, i.e. when the difference between M and the unit matrix of order 4 is an infinitesimal matrix;

(3) a *translation* (which can be interpreted to be a rotation through a zero angle about a straight line at infinity) when we can put

$$M = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so that the matrix equation (1) is equivalent to the three scalar equations

$$x_1 = x + p, \quad y_1 = y + q, \quad z_1 = z + r.$$

In all such cases it is a rigid transformation

It will be shown (see Art. 4) that every rotation about O is a rotation about a straight line through O, and therefore (see Art. 3) a resultant of infinitesimal rotations. Moreover every translation is clearly expressible as a resultant of infinitesimal translations. Hence from Ex. iv below we can conclude that

A rigid transformation is an equitense transformation which is expressible as a resultant of infinitesimal equitense transformations.

Again an equitense transformation will be called one:

about a point A when it leaves the position of A unaltered;

about a straight line (or axis) L when it leaves the position of every point of L unaltered;

about a plane p when it leaves the position of every point of p unaltered.

All equitense transformations about a given finite point (or straight line or plane) clearly constitute a group.

Ex. i. If an equitense transformation in Ω leaves the positions of two different finite points A and B unaltered, it necessarily leaves the position of every point of the straight line AB unaltered, and is an equitense transformation about AB. If it leaves the positions of three non-collinear finite points A, B, C unaltered, it necessarily leaves unaltered the position of every point in the plane ABC, and is an equitense transformation about that plane. If it leaves the positions of three non-coplanar finite points A, B, C, D unaltered, it necessarily leaves unaltered the position of every point of Ω unchanged, and is the identical transformation.

For if it leaves unaltered the points whose co-ordinates are (x, y, z) , (x', y', z') , (x'', y'', z'') , it also leaves unaltered the point whose co-ordinates are

$$(\lambda x + \mu x' + \nu x'', \lambda y + \mu y' + \nu y'', \lambda z + \mu z' + \nu z''), \text{ where } \lambda + \mu + \nu = 1.$$

Ex. ii. A given equitense transformation (1) converts O into the point O' whose co-ordinates with reference to (OX, OY, OZ) are p, q, r ; and it converts OX, OY, OZ into the mutually rectangular axes $O'X', O'Y', O'Z'$ whose direction-cosines with reference to (OX, OY, OZ) are $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$ respectively. The rectangular axes $(O'X', O'Y', O'Z')$ form a right-handed or left-handed set according as $\det \phi = 1$ or $\det \phi = -1$; and they can clearly be any set of rectangular axes drawn from a finite point in Ω .

Since P_1 in (1) can be any finite point of Ω , and the co-ordinates of P_1 with reference to $(O'X', O'Y', O'Z')$ must be the same as those of P with reference to (OX, OY, OZ) , we see by writing

$$(x, y, z) \text{ for } (x_1, y_1, z_1) \text{ and } (x', y', z') \text{ for } (x, y, z)$$

in (1) that if any finite point P of Ω has co-ordinates (x, y, z) with reference to (OX, OY, OZ) , then the co-ordinates (x', y', z') of the same point with reference to $(O'X', O'Y', O'Z')$ are given by the equation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}, \quad (\text{equivalent to } \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \mathbf{M} \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}). \quad (1')$$

which represents a transformation of rectangular co-ordinates. The two sets of rectangular axes (OX, OY, OZ) and $(O'X', O'Y', O'Z')$ are like-handed when and only when $\det \phi = 1$.

Ex. iii. Equitense transformations about a point.

In order that the equitense transformation (1) may be one about the origin O , it is necessary and sufficient that $p = q = r = 0$. The general equations of an equitense transformation about the origin O , about a finite point whose co-ordinates with reference to (OX, OY, OZ) are a, b, c can be expressed in the respective forms

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \begin{bmatrix} x_1 - a \\ y_1 - b \\ z_1 - c \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x - a \\ y - b \\ z - c \end{bmatrix},$$

where in each equation the prefactor on the right is a real square semi-unit matrix.

Ex. iv. The matrix \mathbf{M} in an equitense transformation (1) can be expressed as a product in the forms

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & P \\ 0 & 1 & 0 & Q \\ 0 & 0 & 1 & R \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

where

$$\begin{array}{l} P \\ Q \\ R \end{array} = \begin{array}{c} \left[\begin{array}{ccc} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{array} \right] \left[\begin{array}{c} p \\ q \\ r \end{array} \right] \end{array}$$

Thus any equitense transformation can be regarded as the resultant of an equitense transformation T about O followed by a translation, and also as the resultant of a translation followed by the same equitense transformation T about O .

2. Equitense bodies or configurations ; congruent and pseudo-congruent bodies.

Two finite bodies A and B situated in Ω will be said to be equitense with one another when there exists a one-one correspondence between their points of such a character that the (positive or undirected) distance between any two points of A is equal to the distance between the two corresponding points of B . The following general remarks concerning two such bodies can be established from those properties of equitense transformations which are proved in this paper.

If A and B are 3-dimensional bodies, and are equitense with one another, there exists in general one and only one equitense transformation T in Ω which converts A into B . According as T is a rigid or pseudo-rigid transformation, A and B will be said to be *congruent* or *pseudo-congruent* with another in Ω (alternatively to be *like* or *unlike* in handedness).

If A and B are *congruent* with one another, we can (in many ways) construct a series of mutually congruent bodies $A, C_1, C_2, \dots B$ such that the distance between every pair of corresponding points of any two consecutive bodies of the series is infinitesimal ; and each of these bodies can be converted into the next by an infinitesimal equitense transformation. Thus the equitense transformation converting A into B can be regarded as the resultant of a number of successive infinitesimal equitense transformations ; or A can be converted into B by a continuous displacement in which it moves as a rigid body.

If A and B are *pseudo-congruent* with one another, the equitense transformation converting A into B cannot be regarded as the resultant of infinitesimal equitense transformations ; for such a resultant would necessarily be a rigid transformation. Each of two such bodies may be called an *image* of the other in Ω . If C is any other body in Ω which is equitense with A , then C is congruent with one of the bodies A and B , being convertible into that body by a rigid transformation ; and it is pseudo-congruent with the other body, being convertible into that other body by a pseudo-rigid transformation.

In particular cases as when A and B are two spheres or two regular polyhedra, and are equitense with one another, there exist both rigid and pseudo-rigid transformations converting A into B but the correspondences between the points of A and B will be different in different transformations. When the correspondence has been fixed, there is only one transformation as in the general case.

The above remarks (see Art. 7) are applicable only to 3-dimensional bodies in Ω . If A and B are two 2-dimensional (or 1-dimensional) bodies in Ω which are equitense with one another, there exist many equitense transformations in Ω (both rigid and pseudo-rigid) converting A into B, and we can always regard A and B as being congruent to one another in Ω .

3. Equitense transformations about the origin; rotations and pseudo-rotations; perversions.

$$\text{Let } \phi = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \quad \text{and } \phi^{-1} = \phi' = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \quad (1)$$

be respectively a given real square semi-unit matrix of order 3 and its inverse (or conjugate), which is also a real square semi-unit matrix. Also in matrix equations let

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, P_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, P' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, P_1' = \begin{bmatrix} x'_1 \\ y'_1 \\ z'_1 \end{bmatrix}, \dots \quad (2)$$

Then if (OX, OY, OZ) is a given set of *right-handed* rectangular axes drawn from O, and if these are used as axes of co-ordinates, the *general equation of an equitense transformation about O* in Ω is

$$P_1 = \phi \cdot P, \text{ (equivalent to } P = \phi^{-1} \cdot P) \quad \dots \quad (A)$$

where x, y, z are the co-ordinates of any finite point P with reference to (OX, OY, OZ), and x_1, y_1, z_1 are the co-ordinates with reference to the same axes of the point P_1 in which P is converted by the transformation. If $\det \phi = 1$, the equation (A) represents a rigid transformation about O, which will be called a *rotation* about O; if $\det \phi = -1$, it represents a pseudo-rigid transformation about O, which will be called a *pseudo-rotation* about O.

The transformation converts the rectangular axes OX, OY, OZ into the rectangular axes OX', OY', OZ' whose direction cosines with reference to (OX, OY, OZ) are respectively

$$(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3); \quad \dots \quad (3)$$

and the set of axes (OX', OY', OZ') is right-handed or left-handed according as the transformation is a rotation or a pseudo-rotation. Since OX', OY', OZ' can be any set of rectangular axes drawn from O , and their direction-cosines with reference to (OX, OY, OZ) are known when they are known, we see that there is one and only one equitense transformation about O which converts the axes OX, OY, OZ into another set of rectangular axes drawn from O .

By writing (x, y, z) for (x_1, y_1, z_1) and (x', y', z') for (x, y, z) in (A) we see as in Ex. ii of Art. 1 that the *general equation of a transformation of rectangular co-ordinates* from the axes (OX, OY, OZ) to the axes (OX', OY', OZ') is

$$P' = \phi^{-1} \cdot P, \text{ (equivalent to } P = \phi \cdot P') \quad \dots \quad (B)$$

where x, y, z are the co-ordinates of any point P with reference to (OX, OY, OZ) , and x', y', z' are the co-ordinates of the same point P with reference to (OX', OY', OZ') . The two sets of rectangular axes are like-handed when and only when $\det \phi = 1$.

Now let OX', OY', OZ' be any second set of rectangular axes drawn from O in Ω , not necessarily those mentioned above; let the equation of a transformation of rectangular co-ordinates from (OX, OY, OZ) to (OX', OY', OZ') be

$$P' = \omega^{-1} \cdot P \text{ (equivalent to } P = \omega \cdot P'), \quad \dots \quad (4)$$

where ω and ω^{-1} are real square semi-unit matrices; and let the points P, P_1 in (A) have co-ordinates:

$$(x, y, z), (x_1, y_1, z_1) \text{ with reference to } (OX, OY, OZ);$$

$$(x', y', z'), (x'_1, y'_1, z'_1) \text{ with reference to } (OX', OY', OZ').$$

Then the equation (4) and the corresponding equation

$$P_1' = \omega^{-1} \cdot P_1 \text{ (equivalent to } P_1 = \omega \cdot P_1') \quad \dots \quad (5)$$

show that when OX', OY', OZ' are taken as axes of co-ordinates, the equation of the equitense transformation (A) is

$$P_1' = \omega^{-1} \phi \omega \cdot P', \text{ (equivalent to } P' = \omega^{-1} \phi^{-1} \omega \cdot P_1'). \quad \dots \quad (A')$$

It will be more convenient to express this result in another form. If the equation of a given equitense transformation about O is

$$P_1' = \psi. P' \text{ (equivalent to } P' = \psi^{-1}. P), \quad \dots \quad (C')$$

when OX', OY', OZ' are axes of co-ordinates then the equation of the same equitense transformation is

$$P_1 = \omega\psi\omega^{-1}. P, \text{ (equivalent to } P = \omega\psi\omega^{-1}. P_1), \quad \dots \quad (C)$$

when OX, OY, OZ are axes of co-ordinates.

The equitense transformation (A) in which

$$\phi = \begin{bmatrix} 1, & 0, & 0 & \cos\theta, & 0, & \sin\theta & \cos\theta, & -\sin\theta, & 0 \\ 0, & \cos\theta, & -\sin\theta & 0, & 1, & 0 & \sin\theta, & \cos\theta, & 0 \\ 0, & \sin\theta, & \cos\theta & -\sin\theta, & 0, & \cos\theta & 0, & 0, & 1 \end{bmatrix}$$

are *right-handed rotations through the angle θ about the (right-handed) co-ordinate axes OX, OY, OZ*. Replacing θ by $-\theta$, we obtain the corresponding *left-handed rotations through the angle θ about the same axes*. If $\theta = \theta_1 + \theta_2 + \theta_3 + \dots$, a right-handed rotation through the angle θ about OZ is the resultant of successive right-handed rotations through the angles $\theta_1, \theta_2, \theta_3, \dots$ about OZ. Consequently a rotation about any axis can always be expressed as the resultant of successive infinitesimal rotations about that axis.

The simplest equitense transformations about O are the eight perversions, *viz.* the transformations (A) in which

$$\phi = \begin{bmatrix} \pm 1. & 0, & 0 \\ 0, & \pm 1. & 0 \\ 0, & 0, & +1 \end{bmatrix}$$

They clearly form a complete group. The perversion in which the signs of the successive diagonal elements of ϕ starting from the top are

$$+ \quad +$$

is the *identical transformation*. The perversions in which those signs are

$$- \quad + \quad +, \quad + \quad - \quad +, \quad + \quad + \quad -$$

are respectively reflections about the co-ordinate planes $x=0$, $y=0$, $z=0$, these being pseudo-rotations. The perversions in which those signs

$$+ \quad - \quad - \quad \quad - \quad + \quad \quad - \quad - \quad +$$

are respectively reflections about the co-ordinate axes OX, OY, OZ, and are also rotations (right-handed or left-handed) through two right angles about those axes. The perversion in which those signs are

is a reflection about the origin, which will often be called the *inversion*.

Ex. i. The inversion (or reflexion about O) can be regarded as the resultant of :

- (i) three successive reflexions about three mutually perpendicular planes (OY, OZ), (OZ, OX), (OX, OY) drawn through O;
- (ii) a rotation through two right angles about any axis OZ drawn from O followed by (or preceded by) a reflexion about the plane (OX, OY) drawn through O perpendicular to that axis.

Ex. ii. Equation of a reflection about the plane $lx + my + nz = 0$ when l, m, n are direction-cosines.

Let $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, (l, m, n) be the direction-cosines of three rectangular axes OX', OY', OZ' with reference to the given co-ordinate axes OX, OY, OZ; so that in (4) we have

$$\omega = \begin{vmatrix} \lambda_1 & \lambda_2 & l \\ \mu_1 & \mu_2 & m \\ \nu_1 & \nu_2 & n \end{vmatrix}.$$

When OX', OY', OZ' are axes of co-ordinates, the equation of the given plane is $z' = 0$, and the equation of a reflexion about it is $z'_1 = -z_1$, i.e.

$$P'_1 = \phi \cdot P', \text{ where } \phi = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = I - 2 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

It follows as in (C') and (C) that when OX, OY, OZ are axes of co-ordinates, the equation of a reflexion about the given plane $lx + my + nz = 0$ is

$$P_1 = \psi P, \text{ where } \psi = \omega \phi \omega^{-1}.$$

$$\begin{aligned} \text{i.e. } \psi = I - 2 \quad [l, m, n] = & \begin{vmatrix} 1 - 2l^2 & -2lm & -2ln \\ -2ml & 1 - 2m^2 & -2mn \\ -2nl & -2nm & 1 - 2n^2 \end{vmatrix} \end{aligned}$$

This result clearly remains true when the axes OX, OY, OZ are left-handed.

Ex. iii. Equation of a reflection in the given plane $lx + my + nz + p = 0$ when l, m, n are direction-cosines.

If ψ is the same matrix as in Ex. ii, this equation is

$$\begin{bmatrix} x_1 + lp \\ y_1 + mp \\ z_1 + np \end{bmatrix} = \psi \begin{bmatrix} x + lp \\ y + mp \\ z + np \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-2l^2 & -2lm & -2ln & -2lp \\ -2ml & 1-2m^2 & -2mn & -2mp \\ -2nl & -2nm & 1-2n^2 & -2np \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Ex. iv. If a plane p can be converted into a plane q by a right-handed rotation through an angle $\frac{1}{2}\alpha$ about any axis, then the resultant of two successive reflections in p and q is a right-handed rotation through an angle α about that axis.

This can be seen by taking the given axis to be OZ , and p, q to be the planes

$$y=0, x \sin \frac{1}{2}\alpha - y \cos \frac{1}{2}\alpha = 0.$$

Ex. v. Equation of a right-handed rotation through an angle θ about a given axis OZ' whose direction-cosines are λ, μ, ν .

Let $(\lambda_1, \mu_1, \nu_1), (\lambda_2, \mu_2, \nu_2), (\lambda, \mu, \nu)$ be the direction-cosines of three right-handed rectangular axes OX', OY', OZ' with reference to the given (right-handed rectangular) co-ordinate axes OX, OY, OZ , so that in (4) we have

$$\omega = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda \\ \mu_1 & \mu_2 & \mu \\ \nu_1 & \nu_2 & \nu \end{bmatrix}, \quad \text{and } \det \omega = 1.$$

When OX', OY', OZ' are axes of co-ordinates, the equation of the given rotation is

$$P_1' = \Theta P', \quad \text{where } \Theta = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows as in (C') and (C) that when OX, OY, OZ are axes of co-ordinates, the equation of the given rotation is

$$P_1 = \phi P, \quad \text{where } \phi = \omega \Theta \omega^{-1},$$

$$\text{i.e. } \phi = I + (1 - \cos \theta) \begin{bmatrix} \lambda^2 - 1 & \lambda\mu & \lambda\nu \\ \mu\lambda & \mu^2 - 1 & \mu\nu \\ \nu\lambda & \nu\mu & \nu^2 - 1 \end{bmatrix} + \sin \theta \begin{bmatrix} 0 & -\nu & \mu \\ \nu & 0 & -\lambda \\ -\mu & \lambda & 0 \end{bmatrix}.$$

This result could also have been deduced from Exs. ii and iv by treating the given rotation as the resultant of two successive reflexions about two planes containing OZ' which are inclined to one another at an angle $\frac{1}{2}\theta$.

If OX, OY, OZ were a left-handed set of rectangular axes, the same equation would represent a left-handed rotation through an angle θ about OZ'.

Ex. vi. If in Ex. v we express the square semi-unit matrix ϕ in the form

$$\phi = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix},$$

we have

$$l_1 = 1 + (1 - \cos \theta) \cdot (\lambda^2 - 1) = \lambda^2 + \cos \theta \cdot (1 - \lambda^2),$$

$$m_2 = 1 + (1 - \cos \theta) \cdot (\mu^2 - 1) = \mu^2 + \cos \theta \cdot (1 - \mu^2),$$

$$n_3 = 1 + (1 - \cos \theta) \cdot (\nu^2 - 1) = \nu^2 + \cos \theta \cdot (1 - \nu^2),$$

$$n_2 + m_3 = 2(1 - \cos \theta) \cdot \mu\nu, \quad l_3 + n_1 = 2(1 - \cos \theta) \cdot \nu\lambda, \quad m_1 + l_2 = 2(1 - \cos \theta) \cdot \lambda\mu,$$

$$n_2 - m_3 = 2 \sin \theta \cdot \lambda, \quad l_3 - n_1 = 2 \sin \theta \cdot \mu, \quad m_1 - l_2 = 2 \sin \theta \cdot \nu.$$

4. The axis and angle of a rotation about the origin O whose equation is given in the general form.

Let the equation of a rotation about the origin O of the right-handed rectangular axes OX, OY, OZ be given in the general form

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or } P_1 = \phi \cdot P, \quad \dots \quad (D)$$

where ϕ is a given real square semi-unit matrix whose determinant has the value 1; and let

$$\phi(\rho) = \phi - \rho I = \begin{bmatrix} l_1 - \rho & l_2 & l_3 \\ m_1 & m_2 - \rho & m_3 \\ n_1 & n_2 & n_3 - \rho \end{bmatrix} \quad \dots \quad (1)$$

be the characteristic matrix of ϕ . The latent roots of ϕ are the roots of the equation $\det \phi(\rho) = 0$ or

$$(\rho - 1)\{\rho^2 - (l_1 + m_2 + n_3 - 1)\rho + 1\} = 0, \quad \dots \quad (2)$$

and because 1 is always a latent root, the square matrix $\phi(1)$ is always degenerate. The position of a finite point P whose coordinates with reference to (OX, OY, OZ) are x, y, z will be unaltered by the rotation if and only if $\phi P = P$, i.e., if and only if

$$\phi(1) \cdot P = 0. \quad \dots (3)$$

Because $\phi(1)$ is always degenerate, this equation always has at least one finite non-zero solution for the matrix P ; and if P is the corresponding point, the rotation (see Ex. i of Art 1) is one about the straight line OP .

Thus every rotation about O (which may be any finite point) is a rotation about a straight line passing through O .

In the particular case when (1) is the identical transformation, i.e., when $\phi = I$, it leaves the positions of all points of Ω unaltered, and can be interpreted to be a rotation through an angle 0 about any straight line we please passing through O or about any finite straight line whatever.

To obtain all the points whose positions are unaltered by the rotation, or all the solutions of (3), in other cases, we will consider the conjugate reciprocal (or the reciprocal) of $\phi(1)$, which is the symmetric matrix

$$\Phi = \begin{vmatrix} 1+l_1-m_2-n_3 & l_2+m_1 & l_3+n_1 \\ m_1+l_2 & 1+m_2-n_3-l_1 & m_3+n_2 \\ n_1+l_3 & n_2+m_3 & 1+n_3-l_1-m_2 \end{vmatrix} \quad \dots (4)$$

Because $\phi(1) \cdot \Phi = \det \phi(1) \cdot I = 0$,

the equation (3) is satisfied when x, y, z are proportional to the 1st, 2nd, 3rd elements in any vertical row of Φ . Again because $\phi(1)$ is degenerate, the rank of Φ cannot exceed 1; therefore by one of the properties of symmetric matrices the radicals

$$Q_1 = \sqrt{1+l_1-m_2-n_3}, \quad Q_2 = \sqrt{1+m_2-n_3-l_1}, \\ Q_3 = \sqrt{1+n_3-l_1-m_2} \quad (5)$$

can be so chosen that

$$\Phi = \begin{vmatrix} Q_1 \\ Q_2 \\ Q_3 \end{vmatrix} [Q_1, Q_2, Q_3]. \quad (6)$$

There are then two possible choices of Q_1, Q_2, Q_3 , any one of the radicals which is not 0 having whichever we please of its two possible values.

From the equations

$$\begin{aligned} m_2 n_3 - m_3 n_2 &= l_1, & m_3 n_1 - m_1 n_3 &= l_2, & m_1 n_2 - m_2 n_1 &= l_3, \\ n_2 l_3 - n_3 l_2 &= m_1, & n_3 l_1 - n_1 l_3 &= m_2, & n_1 l_2 - n_2 l_1 &= m_3, \\ l_2 m_3 - l_3 m_2 &= n_1, & l_3 m_1 - l_1 m_3 &= n_2, & l_1 m_2 - l_2 m_1 &= n_3, \end{aligned}$$

it follows that

$$\begin{aligned} (n_2 + m_3)^2 &= (1 + m_2 - n_3 - l_1)(1 + n_3 - l_1 - m_2), \\ (l_3 + n_1)^2 &= (1 + n_3 - l_1 - m_2)(1 + l_1 - m_2 - n_3), \\ (m_1 + l_2)^2 &= (1 + l_1 - m_2 - n_3)(1 + m_2 - n_3 - l_1), \end{aligned} \quad (7)$$

$$\begin{aligned} (n_2 - m_3)^2 &= (1 + l_1 + m_2 + n_3)(1 + l_1 - m_2 - n_3), \\ (l_3 - n_1)^2 &= (1 + l_1 + m_2 + n_3)(1 + m_2 - n_3 - l_1), \\ (m_1 - l_2)^2 &= (1 + l_1 + m_2 + n_3)(1 + n_3 - l_1 - m_2), \end{aligned} \quad (7')$$

$$\begin{aligned} (l_3 - n_1)(m_1 - l_2) &= (1 + l_1 + m_2 + n_3)(n_2 + m_3), \\ (m_1 - l_2)(n_2 - m_3) &= (1 + l_1 + m_2 + n_3)(l_3 + n_1), \\ (n_2 - m_3)(l_3 - n_1) &= (1 + l_1 + m_2 + n_3)(m_1 + l_2). \end{aligned} \quad (7'')$$

The equations (7) and (7') show that the real factors occurring on the right in them are either all positive or all negative. If they were all negative, it would follow that $3 - l_1 - m_2 - n_3$, the sum of the three diagonal elements of Φ in (4), is negative, but this is impossible because l_1, m_2, n_3 are all real quantities whose numerical values do not exceed 1. We conclude that the real quantities

$$1 + l_1 - m_2 - n_3, \quad 1 + m_2 - n_3 - l_1, \quad 1 + n_3 - l_1 - m_2, \quad \dots \quad (8)$$

$$1 + l_1 + m_2 + n_3, \quad 3 - l_1 - m_2 - n_3, \quad \dots \quad (8')$$

are all positive. Hence the three radicals Q_1, Q_2, Q_3 and the two radicals

$$Q = \sqrt{1 + l_1 + m_2 + n_3}, \quad R = \sqrt{3 - l_1 - m_2 - n_3}, \quad (5')$$

which satisfy the equations

$R^2 = Q_1^2 + Q_2^2 + Q_3^2$, $Q^2 + R^2 = 4$, $\frac{1}{2}(Q^2 - R^2) = l_1 + m_2 + n_3 - 1$, (9)
are all real. We have $R=0$ (or $Q_1=Q_2=Q_3=0$) when and only when $l_1=m_2=n_3=1$, i.e., when and only when (D) is the identical transformation.

If Q_1, Q_2, Q_3 are chosen in accordance with (6), we must have

$$n_2 + m_3 = Q_2 Q_3, \quad l_3 + n_1 = Q_3 Q_1, \quad m_1 + l_2 = Q_1 Q_2.$$

The equations (7) are then satisfied, and the equations (7') and (7'') will also be satisfied if and only if

$$n_2 - m_3 = \epsilon Q Q_1, \quad l_3 - n_1 = \epsilon Q Q_2, \quad m_1 - l_2 = \epsilon Q Q_3,$$

where ϵ is either 1 or -1 . Whichever choice has been made of Q_1, Q_2, Q_3 (supposing that they are not all equal to 0), we can choose Q so that $\epsilon = 1$. Consequently we can, and always will, choose the radicals Q, Q_1, Q_2, Q_3 so that

$$Q_2 Q_3 = n_2 + m_3, \quad Q_3 Q_1 = l_3 + n_1, \quad Q_1 Q_2 = m_1 + l_2, \quad \dots \quad (10)$$

$$Q Q_1 = n_2 - m_3, \quad Q Q_2 = l_3 - n_1, \quad Q Q_3 = m_1 - l_2; \quad \dots \quad (10')$$

and the equation (6) is then satisfied. Except when (D) is the identical transformation, there are two and only two possible choices of these four radicals, the signs of all being changed when the sign of any one which does not vanish is changed.

That the three quantities (8) and the quantity $3 - l_1 - m_2 - n_3$ are necessarily positive follows immediately from the fact that Φ is a real symmetric matrix whose rank does not exceed 1; for the diagonal elements of such a matrix must all have the same sign.

We now see that when (D) is not the identical transformation, the diagonal elements of Φ do not all vanish; therefore Φ has rank 1, $\Phi(1)$ has rank 2, and the equation (3) has only one distinct non-zero solution given by

$$x:y:z = Q_1:Q_2:Q_3; \quad \dots \quad (11)$$

therefore (D) is a rotation about the uniquely determinate straight line (11), which is the locus of all points whose positions are unaltered. There are two possible axes OA and OA' , drawn from O in opposite directions along that straight line. After choosing Q_1, Q_2, Q_3, Q in accordance with (10) and (10'), and selecting one of the two possible values of R , we will take OA to be the axis

$$\frac{x}{Q_1} = \frac{y}{Q_2} = \frac{z}{Q_3}$$

with direction-cosines

$$\lambda = \frac{Q_1}{R}, \quad \mu = \frac{Q_2}{R}, \quad \nu = \frac{Q_3}{R}. \quad \dots \quad (12)$$

When (D) is the identical transformation, *i.e.*, when $Q_1=Q_2=Q_3=0$, both $\phi(1)$ and Φ are zero matrices, each having rank 0.

We will next determine the possible angles of rotation.

The three latent roots of ϕ , being the three roots of the equation (3), are

$$1 \text{ and } \frac{1}{2}\{(l_1+m_2+n_3-1) \pm \sqrt{(l_1+m_2+n_3-1)^2-4}\},$$

$$\text{i.e. } 1 \text{ and } \frac{1}{2}\{(l_1+m_2+n_3-1) \pm QR\sqrt{-1}\}.$$

Since Q and R are real, *i.e.*, since $\frac{1}{2}(l_1+m_2+n_3-1)$ is not less than -1 and not greater than 1 , we can always determine a real angle θ such that

$$\cos\theta = \frac{1}{2}(l_1+m_2+n_3-1) = \frac{1}{2}(Q^2-R^2) = \frac{1}{2}Q^2-1 = 1-\frac{1}{2}R^2; \quad \dots \quad (13)$$

and if $i = \sqrt{-1}$, the three latent roots are then

$$1, \cos\theta + i\sin\theta, \cos\theta - i\sin\theta. \quad \dots \quad (14)$$

This result is in accordance with the facts that every real latent root of a real square semi-unit matrix must be either 1 or -1 , and that the latent roots which are not real must occur in pairs of the form $\cos\theta \pm i\sin\theta$, where θ is a real angle. The latent roots (14) are the same for all values of θ satisfying the equation (13). If α is any one angle satisfying the two mutually consistent equations

$$\cos\alpha = \frac{1}{2}(l_1+m_2+n_3-1), \sin\alpha = \frac{1}{2}QR, \quad \dots \quad (15)$$

or any one angle satisfying the two mutually consistent equations

$$2\cos\frac{1}{2}\alpha = \epsilon Q, 2\sin\frac{1}{2}\alpha = \epsilon R, \quad \dots \quad (15')$$

where ϵ is either 1 or -1 , then $\theta = \alpha$ and $\theta = -\alpha$ are solutions of (13), and every other solution differs from one of these two by a multiple of 2π .

The three latent roots of ϕ are all real in two particular cases only, *viz.*,

$$(i) \text{ when } R=0, \text{ i.e. when } l_1+m_2+n_3=3;$$

$$(ii) \text{ when } Q=0, \text{ i.e. when } l_1+m_2+n_3=-1.$$

The first particular case occurs when and only when $l_1=m_2=n_3=1$, *i.e.*, when and only when (D) is the identical transformation; the three latent roots being then $1, 1, 1$; the angle θ being 0 or a multiple of 2π ; and Q_1, Q_2, Q_3 being all 0 . The second particular case occurs when and only when the semi-unit matrix ϕ is symmetric but not a unit

matrix, as we see from the equations (7'). The angle θ is then an odd multiple of π ; the three latent roots of ϕ are 1, -1 , -1 ; and the rotation is not the identical transformation. In all other cases there are two distinct values of θ , if values differing by a multiple of 2π are not considered to be distinct from one another.

By a well known property of real square semi-unit matrices, the equation (D) can be converted by a transformation of rectangular axes into an equation of the form

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \dots \quad (D')$$

where θ is any solution of the equation (13); and (D') represents a rotation through an angle θ about the new axis of z , which when $\phi \neq 1$ must be one of the two axes OA and OA'. We will verify this without using the general theory of semi-unit matrices, and at the same time determine what angles of rotation are appropriate to each axis.

Supposing that $\phi \neq 1$, let OA be the axis (12), and let α be any solution of the two equations (15) or the two equations (15'). Then from Ex. v of Art. 3 we see that the equations of the right-handed rotations $\pm\alpha$ about OA are respectively

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \left\{ \frac{1}{2} \begin{bmatrix} 2l_1 & l_2 + m_1 & l_3 + n_1 \\ m_1 + l_2 & 2m_2 & m_3 + n_2 \\ n_1 + l_3 & n_2 + m_3 & 2n_3 \end{bmatrix} \right. \\ \left. \pm \begin{bmatrix} 0 & l_2 - m_1 & l_3 - n_1 \\ m_1 - l_2 & 0 & m_3 - n_2 \\ n_1 - l_3 & n_2 - m_3 & 0 \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

the equation with the upper sign being the equation (D). Accordingly we have the following theorem in which the axes OX, OY, OZ are right-handed.

Theorem I. *When $\phi=1$, the equation (D) represents the identical transformation. In all other cases it represents a right-handed rotation about an axis OA whose direction-cosines are*

$$\lambda = \frac{Q_1}{R}, \quad \mu = \frac{Q_2}{R}, \quad \nu = \frac{Q_3}{R}$$

and through an angle α which is determined (uniquely except for an arbitrary additive multiple of 2π) by the equations

$$\cos \alpha = \frac{1}{2}(l_1 + m_2 + n_3 - 1), \quad \sin \alpha = \frac{1}{2}QR.$$

It is immaterial which sign we ascribe to the radical R , and which of the two possible sets of signs we ascribe to the radicals Q_1, Q_2, Q_3, Q . We can choose Q and R to be both positive, and consider α to be the real angle lying between 0 and π which is uniquely determined by the two equations

$$2\cos \frac{1}{2}\alpha = Q, \quad 2\sin \frac{1}{2}\alpha = R.$$

5. The pseudo-axis and angle of a pseudo-rotation about the origin O whose equation is given in the general form.

Let the equation of a pseudo-rotation about the origin O of the right-handed rectangular axes OX, OY, OZ be given in the general form

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or } P_1 = \phi \cdot P \quad \dots \quad (E)$$

where ϕ is a given real square semi-unit matrix whose determinant has the value -1 . The quantities $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$ are the direction-cosines with reference to (OX, OY, OZ) of the left-handed rectangular axes OX', OY', OZ' into which OX, OY, OZ are converted by the pseudo-rotation. Putting $\psi = -\phi$, we can interpret the transformation (E) by comparing it with the transformation

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} -l_1 & -l_2 & -l_3 \\ -m_1 & -m_2 & -m_3 \\ -n_1 & -n_2 & -n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{or } P_1 = \psi \cdot P \quad \dots \quad (E')$$

which is a rotation about O . Let (E') be a right-handed (or left-handed) rotation about an axis OA through an angle θ' , and let

$$\theta = \theta' \pm \pi.$$

Then (E) is the resultant of

- a right-handed (or left-handed) rotation through an angle θ' about OA ;
- an inversion, or reflexion about the origin O ;

the order in which the two operations are performed being immaterial. Therefore by Ex. *i* of Art 3, it can also be regarded as the resultant of :

- a right-handed (or left-handed) rotation through an angle θ about OA ;
- a reflexion about the plane through O perpendicular to OA ;

the order in which these two operations are performed being again immaterial. Taking the latter view, we will describe (E) as a right-handed (or left-handed) pseudo-rotation of *angle* θ having OA as a *pseudo-axis* ; a pseudo-axis being a locus of points which suffer a reflexion about the origin.

If $\theta=0$ or is a multiple of 2π , the pseudo-rotation (E) is simply a reflexion about the plane through O perpendicular to OA.

If $\theta=\pm\pi$ or is an odd multiple of π , the pseudo-rotation (E) is the inversion, *i.e.*, a reflexion about the origin O : and every straight line drawn from O can be regarded as a pseudo-axis. This case occurs when and only when, $\phi=-1$.

Except when $\phi=-1$, there are two and only two pseudo-axes OA and OA', drawn from O in opposite directions.

Applying Art 4 to the rotation (E'), we see that the radicals

$$Q_1 = \sqrt{-1 + l_1 - m_2 - n_3}, \quad Q_2 = \sqrt{-1 + m_2 - n_3 - l_1},$$

$$Q_3 = \sqrt{-1 + n_3 - l_1 - m_2}, \quad \dots \quad (1)$$

$$Q = \sqrt{-1 + l_1 + m_2 + n_3}, \quad R = \sqrt{-3 - l_1 - m_2 - n_3} \quad \dots \quad (1')$$

satisfying the equations

$$R^2 = Q_1^2 + Q_2^2 + Q_3^2, \quad Q^2 + R^2 = -4, \quad \frac{1}{2}(Q^2 - R^2) = l_1 + m_2 + n_3 + 1, \quad (2)$$

are all purely imaginary, and that Q_1, Q_2, Q_3, Q can be so chosen as to satisfy the relations

$$Q_2 Q_3 = n_2 + m_3, \quad Q_3 Q_1 = l_3 + n_1, \quad Q_1 Q_2 = m_1 + l_2, \quad \dots \quad (3)$$

$$Q Q_1 = n_2 - m_3, \quad Q Q_2 = l_3 - n_1, \quad Q Q_3 = m_1 - l_2. \quad \dots \quad (3')$$

We have $R=0$ (or $Q_1=Q_2=Q_3=0$) when and only when (E) is the inversion, *i.e.*, $\phi=-1$; and in all other cases there are two and only two possible choices of the four radicals Q_1, Q_2, Q_3, Q consistent with (3) and (3').

When the above defined radicals have been chosen in accordance with (3) and (3') we can deduce from Theorem I the following theorem, in which the axes OX, OY, OZ are right-handed :

Theorem II. When $\phi = -I$, the equation (E) represents the inversion, i.e., a reflexion about the origin O. In all other cases it represents a right-handed pseudo-rotation about O having a pseudo-axis OA whose direction-cosines are

$$\lambda = \frac{Q_1}{R}, \quad \mu = \frac{Q_2}{R}, \quad \nu = \frac{Q_3}{R},$$

and angle α which is determined by the equations

$$\cos \alpha = \frac{1}{2}(l_1 + m_2 + n_3 + 1), \quad \sin \alpha = \frac{1}{2}QR;$$

i.e., it is the resultant of a right-handed rotation α about OA and a reflexion about the plane through O perpendicular to OA.

Ex. i. When the square semi-unit matrix ϕ in (E) has the values

$$\begin{bmatrix} -1, & 0, & 0 \\ 0, & \cos \theta, & -\sin \theta \\ 0, & \sin \theta, & \cos \theta \end{bmatrix}, \quad \begin{bmatrix} \cos \theta, & 0, & \sin \theta \\ 0, & -1, & 0 \\ -\sin \theta, & 0, & \cos \theta \end{bmatrix},$$

$$\begin{bmatrix} \cos \theta, & -\sin \theta, & 0 \\ \sin \theta, & \cos \theta, & 0 \\ 0, & 0, & -1 \end{bmatrix},$$

the equation (E) represents right-handed pseudo-rotations of angle θ about O having OX, OY, OZ respectively as pseudo-axes.

Ex. ii. By a transformation of right-handed rectangular axes it can be deduced from Ex. i that the equation of a right-handed pseudo-rotation of angle θ about O having a pseudo-axis OZ' with direction-cosines λ, μ, ν is

$$P_1 = \phi P,$$

where

$$\phi = -I - (1 + \cos \theta) \begin{bmatrix} \lambda^2 - 1, & \lambda\mu, & \lambda\nu \\ \mu\lambda, & \mu^2 - 1, & \mu\nu \\ \nu\lambda, & \nu\mu, & \nu^2 - 1 \end{bmatrix} + \sin \theta \begin{bmatrix} 0, & -\nu, & \mu \\ \nu, & 0, & -\lambda \\ -\mu, & \lambda, & 0 \end{bmatrix}.$$

Ex. iii. If in Ex. ii we put

$$\phi = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}, \quad \delta = \det \phi = -1,$$

we have

$$l_1 = \delta + (\delta - \cos\theta) \cdot (\lambda^2 - 1) = \delta\lambda^2 + \cos\theta \cdot (1 - \lambda^2),$$

$$m_2 = \delta + (\delta - \cos\theta) \cdot (\mu^2 - 1) = \delta\mu^2 + \cos\theta \cdot (1 - \mu^2),$$

$$n_3 = \delta + (\delta - \cos\theta) \cdot (\nu^2 - 1) = \delta\nu^2 + \cos\theta \cdot (1 - \nu^2),$$

$$n_2 + m_2 = 2(\delta - \cos\theta) \cdot \mu\nu, \quad l_2 + n_1 = 2(\delta - \cos\theta) \cdot \nu\lambda, \quad m_1 + l_2 = 2(\delta - \cos\theta) \cdot \lambda\mu$$

$$n_2 - m_3 = 2\sin\theta \cdot \lambda, \quad l_2 - n_1 = 2\sin\theta \cdot \mu, \quad m_1 - l_2 = 2\sin\theta \cdot \nu$$

The corresponding formulae given in Ex. vi of Art 3 are obtained by putting $\delta = 1$.

6. Analogies between rotations and pseudo-rotations about the origin.

If we make use of Exs. ii and iii of Art. 5, we can give a direct proof of Theorem II which is strictly analogous to that of Theorem I given in Art. 3, provided that we determine those points which suffer a reflexion about the origin O instead of those points whose positions are unaltered.

If $\delta = \det\phi = \pm 1$, then in both theorems we have

$$Q_1 = \sqrt{\delta + l_1 - m_2 - n_3}, \quad Q_2 = \sqrt{\delta + m_2 - n_3 - l_1}, \quad Q_3 = \sqrt{\delta + n_3 - l_1 - m_2},$$

$$Q = \sqrt{\delta + l_1 + m_2 + n_3}, \quad R = \sqrt{3\delta - l_1 - m_2 - n_3},$$

$$Q_2 Q_3 = n_2 + m_3, \quad Q_3 Q_1 = l_3 + n_1, \quad Q_1 Q_2 = n_1 + l_2,$$

$$QQ_1 = n_2 - m_3, \quad QQ_2 = l_3 - n_1, \quad QQ_3 = m_1 - l_2.$$

In both theorems the latent roots of ϕ are the roots in ρ of the equation

$$(\rho - \delta)\{\rho^3 - (l_1 + m_2 + n_3)\rho + 1\} = 0;$$

and both theorems can be proved by determining all the solutions of the matrix equation

$$\phi(\delta) \cdot P = 0.$$

In both theorems the matrix $\phi(\delta)$ is degenerate, and its conjugate reciprocal Φ is the symmetric matrix

$$\Phi = \begin{bmatrix} \delta + l_1 - m_2 - n_3, & l_2 + n_1, & l_3 + n_1 \\ m_1 + l_2, & \delta + m_2 - n_3 - l_1, & m_3 + n_2 \\ n_1 + l_3, & n_2 + m_3, & \delta + n_3 - l_1 - m_2 \end{bmatrix}$$

$$= \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} [Q_1, Q_2, Q_3]$$

Ex. i. From Ex ii of Art. 3 it will be seen that in both theorems we can put

$$\phi = \delta \left\{ 1 - 2 \begin{vmatrix} \lambda_2 \\ \mu_2 \\ \nu_2 \end{vmatrix} [\lambda_2, \mu_2, \nu_2] \right\} \left\{ 1 - 2 \begin{vmatrix} \lambda_1 \\ \mu_1 \\ \nu_1 \end{vmatrix} [\lambda_1, \mu_1, \nu_1] \right\},$$

where $(\lambda_1, \mu_1, \nu_1)$ and $(\lambda_2, \mu_2, \nu_2)$ are two sets of direction-cosines. We then have

$$[Q_1, Q_2, Q_3 Q] = 2 \sqrt{\delta} \cdot [\mu_1 \nu_2 - \mu_2 \nu_1, \nu_1 \lambda_2 - \nu_2 \lambda_1, \lambda_1 \mu_2 - \lambda_2 \mu_1, \lambda_1 \lambda_2 + \mu_1 \mu_2 + \nu_1 \nu_2]$$

$$R = \pm 2 \sqrt{\delta} \cdot \sqrt{(\mu_1 \nu_2 - \mu_2 \nu_1)^2 + (\nu_1 \lambda_2 - \nu_2 \lambda_1)^2 + (\lambda_1 \mu_2 - \lambda_2 \mu_1)^2},$$

$$\cos \alpha = \frac{1}{2} \delta \cdot (Q^2 - R^2); \quad \sin \alpha = \frac{1}{2} QR, \quad Q^2 + R^2 = 4\delta.$$

Ex. ii. The complete axis and complete pseudo-axis of an equitense transformation about the origin O.

If we define the *complete axis* to be the locus of all points whose positions are unaltered by the transformation, and the *complete pseudo-axis* to be the locus of all points which suffer a reflexion about O, the general characters of these two loci for any equitense transformation about O are as shown below, where ;

α means an angle which is not a multiple of π ;

L means a straight line drawn through O perpendicular to μ

p means a plane drawn through O perpendicular to L.

Rotation of angle θ about O.	Complete axis.	Complete pseudo-axis.
$\theta = \alpha$: (ordinary case.)	a st. line L.	The point O.
$\theta = \pi$: (reflexion about a st. line)	a st. line L.	a plane μ .
$\theta = 0$: (the identical transformation)	The 3-way space Ω	The point O.

Pseudo-rotation of angle θ about O.	Complete axis.	Complete pseudo-axis.
$\theta = \alpha$: (ordinary case).	The point O.	a st. line L.
$\theta = 0$: (reflexion about a plane).	a plane p .	a st. line L.
$\theta = \pi$: (reflexion about O).	The point O.	The 3-way space Ω .

According as the complete axis is

the point O, a st. line through O, a plane through O, the 3-way space Ω , the equitense transformation can be regarded as the resultant of successive reflexions about

3, 2, 1, 0 planes through O.

It is a rotation in the second and fourth cases, and a pseudo-rotation in the first and third cases.

7. Pseudo-rigid transformations in Ω_4 regarded as rigid transformations in Ω_5 .

The principles and methods explained in the foregoing articles can be extended to ordinary metrical space Ω_{n+1} of n dimensions, where n is any positive integer. Any particular set of n rectangular axes drawn from a finite point O in Ω_{n+1} can be regarded as right-handed; and by reversing the direction of one of the axes we obtain a second set of rectangular axes in Ω_{n+1} which is left-handed. The choice of a standard set of right-handed rectangular axes in Ω_{n+1} will depend on some standard n -dimensional configuration in Ω_{n+1} .

Let Ω_4 be the ordinary 3-way space Ω which has hitherto been considered, (OX, OY, OZ) being a set of three rectangular axes in Ω_4 ; and let (OX, OY, OZ, OW) be a set of four rectangular axes in an ordinary 4-way metrical space Ω_5 which contains Ω_4 . Then if the first of the equations

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & a \\ m_1 & m_2 & m_3 & b \\ n_1 & n_2 & n_3 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \\ 1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & 0 & 0 \\ m_1 & m_2 & m_3 & 0 & 0 \\ n_1 & n_2 & n_3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \\ 1 \end{bmatrix} \quad \dots (1)$$

represents a pseudo-rigid transformation in Ω_4 , the second equation represents a rigid transformation in Ω_5 . Moreover if S is any 3-dimensional body in Ω_4 which is converted into S_1 by the first transformation,

w being 0 for all points of S , then the second transformation also converts S into S_1 . Consequently S can be converted into S_1 by a succession of infinitesimal equitense transformations in Ω_5 , *i.e.* by a continuous displacement in which it moves as a rigid body; but in general it will be entirely outside the space Ω_1 in all the intermediate positions.

Again of the first of the equations

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 & 0 \\ m_1 & m_2 & m_3 & 0 \\ n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \dots \quad (2)$$

represents a pseudo-rotation in Ω_4 about O , the second equation represents a rotation in Ω_5 about O ; and if S is any 3-dimensional body in Ω_4 which is converted into S_1 by the pseudo-rotation in Ω_4 , then S will also be converted into S_1 by the rotation in Ω_5 represented by the second equation. The two transformations have the same complete axis, but the complete pseudo-axis of the second transformation is the space determined by the complete pseudo-axis of the first transformation and the new co-ordinate axis OW . The pairs of transformations next considered are representative cases of (2).

The two equitense transformations about O represented by the equations

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \dots \quad (3)$$

are equivalent when applied to a 3-dimensional body in Ω_4 . The first is a pseudo-rotation of angle θ in Ω_4 having OZ as a pseudo-axis; the second is a rotation in Ω_3 which is the resultant of a rotation θ about the plane (OZ, OW) and a rotation π about the plane (OX, OY). In the ordinary case when θ is not a multiple of π , the first transformation has the straight line OZ as complete pseudo-axis, the point O as complete axis, whilst the second transformation has

the plane (OZ, OW) as complete pseudo-axis, the point O as complete axis.

In the particular case when $\theta = \pi$ (or is an odd multiple of π), the first of the transformations (3) is a reflexion about O in Ω_4 ; and the second is a reflexion about O in Ω_3 , which is also the resultant of a rotation π about the plane (OZ, OW) and a rotation π about the plane (OX, OY). The first transformation has

the 3-way space Ω_3 as complete pseudo-axis, the point O as complete axis; whilst the second transformation has

the 4-way space Ω_4 as complete pseudo-axis, the point O as complete axis.

In the particular case when $\theta = 0$ (or is a multiple of 2π), the two transformations (3) become

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \dots \quad (4)$$

The first transformation is a reflexion in Ω_4 about the plane (OX, OY); and the second is a rotation in Ω_3 through two right angles about the the plane (OX, OY). The first transformation has

the straight line OZ as complete pseudo-axis, the plane (OX,OY) as
complete axis ;

whilst the second transformation has

the plane (OZ, OW) as complete pseudo-axis, the plane (OX, OY)
as complete axis.

From (4) it will be seen that, so far as events in Ω_4 are concerned, a reflexion in Ω_4 about a plane p is equivalent to a rotation through two right angles about p in Ω_3 . Thus if a man existing in Ω_4 and occupying the configuration H surveys his image H' formed by reflexion about a plane mirror, he will know that he could be carried from H to H' by a rotation in Ω_3 about the plane of the mirror through two right angles, *i.e.*, by a continuous rigid movement in Ω_3 ; but in the execution of this movement his Ω_4 -existence would cease in all the configurations intermediate between H and H'; *i.e.*, in every such intermediate position he would be entirely outside the space Ω_4 to which his existence is confined. Using definitions appropriate to Ω_4 , the right arm of H will be converted into the left arm of H'; but if we used definitions appropriate to Ω_3 , the right arm of H would be converted into the right arm of H'. In the latter case we regard H and H' as two different aspects of the same 3-dimensional entity, and distinguish between the two sides of that entity in Ω_3 , a distinction which is impossible in Ω_4 . Similarly in speaking of the right-hand and left-hand edges of a printed page, we use definitions appropriate to 2-way space; but in order to speak of the right-hand and left-hand edges of a printed leaf, we should have to distinguish between the front page and back page, and use definitions appropriate to 3-way space.

The distinction between congruence and pseudo-congruence (or between a right-handed and left-handed set of three rectangular axes) which occurs in Ω_1 , disappears in Ω_3 ; and both the spaces Ω_4 and Ω_3 have definitions of right-handedness and left-handedness which are peculiar to them. Similarly the distinction between congruence and pseudo-congruence which occurs in Ω_{n+1} , disappears in Ω_{n+3} ; and every space Ω_{n+1} has definitions of right-handedness and left-handedness which are peculiar to it.

ON THE EVALUATION OF SOME FACTORABLE CONTINUANTS.

Part II.

By

SATISH CHANDRA CHAKRABARTI, M.Sc.

In the first part of this paper, published in the *Bulletin of the Calcutta Mathematical Society*, Vol. XIII, Nos. 1 and 2, pp. 71-84 towards the end of the Article 7, there have been given some operations for evaluating a factorable continuant. In combining these operations we have got some identities which are treated in Arts. 3 and 4 of the present paper. In the paper¹ of Mr. Haripada Datta in which the above continuant occurs, there has been given another factorable continuant which has been evaluated in Art. 1 determinantly. In combining the operations given in Art. 1, we have got some more identities which have been established in Arts. 5, 6 and 7. In Art. 2 we have considered the general case of the identities which occur in Arts. 1, 2 and 3 of the first part.

1. The continuant

$$\begin{array}{ccccccc}
 1 & & & & & & \\
 1+a^{n-1}y & 1+y & & & & & \\
 (a-1)(a^{n-1}-1)y & 1+ay & & & & & \\
 (1+y)(1+a^ny)a & 1+a^ny & & & & & \\
 & \dots & \dots & \dots & \dots & & \\
 (a^{n-1}-1)(a-1)ya^{2^{n-1}} & 1+a^{2^{n-1}}y & & & & & \\
 & & & & & & (1+a^{n-2}y)a^{n-1} \quad 1 \quad 2n \\
 = \{ (1+y)(1+ay)(1+a^2y) \dots (1+a^{2^{n-1}}y) \} \\
 \times \{ (1-x)(1-ax)(1-a^2x) \dots (1-a^{n-1}x) \}
 \end{array}$$

¹ Haripada Datta, "On the Failure of Heilermann's Theorem," *Proc. Edin. Math. Soc.* Vol. 35, part 2, 1916-17 or *Univ. Edin. Math. Depart. Session 1917. Research paper No. 7*, pp. 10.

Here the elements, except the first and the last, of the lower minor diagonal, are given by $e_{s,m} = (1+a^{m-2}y)(1+a^{m+n-2}y)a^{m-1}$ and $e_{s,m+1} = (a^m-1)(a^{n-m}-1)ya^{s-m-2}$, where e_r denotes the element of this diagonal in the r th row.

Let us first consider the particular case when $n=3$, viz.

$$\begin{vmatrix} 1 & x & & & \\ 1+a^2y & 1+y, & x & & \\ (a-1)(a^2-1)y, & 1+ay, & & x & \\ & (1+y)(1+a^2y)a, & 1+a^2y, & & x \\ & & (a^2-1)(a-1)ya^2, & 1+a^2y, & x \\ & & & (1+ay)a^2, & 1 \end{vmatrix}$$

On this perform the first operation

$$-a^2(a^2-1)(a-1)y^2(1+y)(1+ay)\text{col}_5 + a^2(a^2-1)(a-1)y^2(1+y)\text{col}_3 \\ + a(a^2-1)y(1+y)\text{col}_4 - (a^2-1)y\text{col}_3 - \text{col}_2 + \text{col}_1$$

This enables us to remove $1-x$ from the last column and then subtracting the first column from the last we can remove another factor $-(1+y)$ from that column and write the co-factor in the form

$$\frac{1}{a^2(a^2-1)(a-1)y^2(1+y)} \times \begin{vmatrix} 1 & x & 0 & 0 & 0 \\ 1+a^2y, & 1+y, & x & 0 & 1 \\ (a-1)(a^2-1)y, & 1+ay, & & x & a(a^2-1)y \\ & (1+y)(1+a^2y)a, & 1+a^2y, & & a^2(a^2-1)(a-1)y^2 \\ & & & (a^2-1)(a-1)ya^2, & -a^2(a^2-1)(a-1)y^2(1+ay) \end{vmatrix}$$

On this determinant perform the second operation

$$(a-1)\text{col}_5 + a^2(a-1)y(1+ay)\text{col}_4 - a^2(a-1)y\text{col}_3 - a\text{col}_2 + \text{col}_1$$

This enables us to remove the factor $(1-ax)$ from the last column and then subtracting the first column from the last we can remove another factor $-(1+ay)$ and get the co-factor in the form

$$\frac{1+a^2y}{y(a-1)} \cdot \begin{vmatrix} 1 & x & \\ 1+a^2y, & 1+y, & 1 \\ (a-1)(a^2-1)y & a^2(a-1)y & \end{vmatrix}$$

On this performing the third operation

$(a^2-1)\text{col}_3 - a^2\text{col}_2 + \text{col}_1$, we have

$$y(a-1)(a^2-1) \begin{vmatrix} 1+a^2y & 1 & x & 1-a^2x \\ 1+a^2y & 1+y & 0 & 0 \end{vmatrix} = (1-a^2x)(1+a^2y) \times (1+a^3y)$$

\therefore the continuant of the 6th order $= (1-x)(1-ax)(1-a^2x)(1+y)$

$$(1+ay)(1+a^2y)(1+a^3y)$$

In the general case if m_k denotes the multiplier of the k th column and l that of the last column, then we have :

In the first operation

$$\left\{ \begin{aligned} m_{2r} &= (-1)^r a^{\frac{3r^2-7r+4}{2}} y^{r-1} \{ (a^{n-1}-1)(a^{n-2}-1) \dots (a^{n-r+1}-1) \} \\ &\quad \{ (1+y)(1+ay) \dots (1+a^{r-2}y) \} \\ m_{2r+1} &= (-1)^r a^{\frac{3r^2-3r}{2}} y^r \{ (a^{n-1}-1)(a^{n-2}-1) \dots (a^{n-r}-1) \} \\ &\quad \{ (1+y)(1+ay) \dots (1+a^{r-2}y) \} \end{aligned} \right.$$

l is governed by these two rules.

In the second operation

$$\left\{ \begin{aligned} m_{2r} &= (-1)^r a^{\frac{3r^2-5r+4}{2}} y^{r-1} \{ (a^{n-2}-1)(a^{n-3}-1) \dots (a^{n-r}-1) \} \\ &\quad \{ (1+ay)(1+a^2y) \dots (1+a^{r-1}y) \} \\ m_{2r+1} &= (-1)^r a^{\frac{3r^2-r+2}{2}} y^r \{ (a^{n-2}-1)(a^{n-3}-1) \dots (a^{n-r-1}-1) \} \\ &\quad \{ (1+ay)(1+a^2y) \dots (1+a^{r-1}y) \} \end{aligned} \right.$$

$$l = a-1.$$

In the third operation

$$\left\{ \begin{aligned} m_{2r} &= (-1)^r a^{\frac{3r^2-3r+4}{2}} y^{r-1} \{ (a^{n-3}-1)(a^{n-4}-1) \dots (a^{n-r-1}-1) \} \\ &\quad \{ (1+a^2y)(1+a^3y) \dots (1+a^ry) \} \\ m_{2r+1} &= (-1)^r a^{\frac{3r^2+r+4}{2}} y^r \{ (a^{n-3}-1)(a^{n-4}-1) \dots (a^{n-r-2}-1) \} \\ &\quad \{ (1+a^2y)(1+a^3y) \dots (1+a^ry) \} \end{aligned} \right.$$

$$l = a^2-1.$$

In the fourth operation

$$\left\{ \begin{aligned} m_{2r} &= (-1)^r a^{\frac{3r^2-r+4}{2}} y^{r-1} \{ (a^{n-1}-1)(a^{n-2}-1)\dots(a^{n-r+2}-1) \} \\ &\quad \{ (1+a^3y)(1+a^4y)\dots(1+a^{r+1}y) \} \\ m_{2r+1} &= (-1)^r a^{\frac{3r^2+3r+6}{2}} y^r \{ (a^{n-1}-1)(a^{n-2}-1)\dots(a^{n-r+1}-1) \} \\ &\quad \{ (1+a^3y)(1+a^4y)\dots(1+a^{r+1}y) \} \\ l &= a^3 - 1. \end{aligned} \right.$$

and so on.

In each of these operations m_1 is always unity. After each operation being performed we shall find a factor of the form $(1-a^r x)$ removable from the last column. Removing this factor and subtracting the first column from the last we shall find another factor of the form $-(1+a^r y)$ removable from the last column. On removing this factor we shall have the co-factor in the form of a determinant on which the next operation is to be performed.

$$\begin{aligned} 2. \quad & \{x(x+1)(x+2) \dots (x+r-1)\} \\ &= \{(x-\delta)(x-\delta-1)(x-\delta-2)\dots(x-\delta-r+1) \\ &+ (\delta+r-1)C_1 \{ (x-\delta)(x-\delta-1)\dots(x-\delta-r+2) \} \\ &+ (\delta+r-1)(\delta+r-2)C_2 \{ (x-\delta)(x-\delta-1)\dots(x-\delta-r+3) \} \\ &+ \dots + \{ (\delta+r-1)(\delta+r-2)(\delta+r-3)\dots(\delta+1) \} C_{r-1} (x-\delta) \\ &+ \{ (\delta+r-1)(\delta+r-2)\dots(\delta+1)\delta \} \text{ identically.} \quad \dots (1) \end{aligned}$$

Proof. If we substitute any of the values $0, -1, -2, \dots, -(r-1)$ for x in (1), then by means of difference formulæ we can show that in each case of these substitutions the left-hand-side expression = the right-hand-side expression = 0. Again if $x = \delta$, each of the two expressions is equal to the last term of the right-hand-side expression. Thus for more than r values of x , the equation (1) is satisfied. Hence it is an identity.

Ex. 1. Putting $2x = x+1$ and $2\delta = \delta+1$ in (1), we get as a particular case of the theorem (1), the same identity as given in Art 1 of the first part.

Ex. 2. Putting $2\delta=1$, $r=h$ and $2x=a+2h-1$ in (1) we have

$$\begin{aligned} & \{(a+2h-1)(a+2h+1)(a+2h+3)\dots(a+4h-3)\} \\ & \quad = \{(a+2h-2)(a+2h-4)\dots(a+2)a\} \\ & \quad + (2h-1)C_1^h \{(a+2h-2)(a+2h-4)\dots(a+2)\} \\ & \quad + (2h-1)(2h-3)C_2^h \{(a+2h-2)(a+2h-4)\dots(a+4)\} + \dots \\ & \quad + \{(2h-1)(2h-3)\dots(2h-2k+1)\}C_k^h \{(a+2h-2)(a+2h-4)\dots(a+2k)\} \\ & \quad + \dots + \{(2h-1)(2h-3)\dots 3\}C_{h-1}^h (a+2h-2) \\ & \quad + \{(2h-1)(2h-3)\dots 3 \cdot 1\}C_h^h \dots \quad (2) \end{aligned}$$

But

$$\begin{aligned} & \{(2h-1)(2h-3)\dots(2h-2k+1)\}C_k^h \\ & \quad = \frac{h}{h-k} \frac{h}{k} \{(2h-1)(2h-3)\dots(2h-2k+1)\} \\ & \quad = \frac{\{h(h-1)(h-2)\dots(h-k+1)\}}{k} \{(2h-1)(2h-3)\dots(2h-2k+1)\} \\ & \quad = \frac{\{2h(2h-2)(2h-4)\dots(2h-2k+2)\}}{2 \cdot 4 \cdot 6 \dots (2k)} \{(2h-1)(2h-3)\dots(2h-2k+1)\} \\ & \quad = \frac{2h}{2h-2k} \{2 \cdot 4 \cdot 6 \dots (2k)\} = \{1 \cdot 3 \cdot 5 \dots (2k-1)\} \frac{2h}{2h-2k} \frac{1}{2k} \\ & \quad = \{1 \cdot 3 \cdot 5 \dots (2k-1)\} C_{2h-2k}^{2h} \end{aligned}$$

Hence from (2) we have

$$\begin{aligned} & \{(a+2h-1)(a+2h+1)\dots(a+4h-3)\} \\ & \quad = \{a(a+2)(a+4)\dots(a+2h-2)\}C_{2h}^{2h} \\ & \quad + 1 \cdot C_{2h-2}^{2h} \{(a+2)(a+4)\dots(a+2h-2)\} \\ & \quad + 1 \cdot 3 C_{2h-4}^{2h} \{(a+4)(a+6)\dots(a+2h-2)\} + \dots \\ & \quad + \{1 \cdot 3 \cdot 5 \dots (2k-1)C_{2h-2k}^{2h} \{a+2k)(a+2k+2)\dots(a+2h-2)\} \\ & \quad + \dots + \{1 \cdot 3 \cdot 5 \dots (2h-3)C_2^{2h} (a+2h-2) + \{1 \cdot 3 \cdot 5 \dots (2h-1)C_0^{2h} \end{aligned}$$

which is the same identity as given in Art. 2 of the first part.

Ex. 3. Similarly by putting $2\delta=3, r=h-1$ and $2x=a+2h-1$ in the theorem (1) we can show that

$$\begin{aligned} \{(a+2h-1)(a+2h+1)\dots(a+4h-5)\} &= \{a(a+2)\dots(a+2h-4)\} \\ &\quad + 1 \cdot C_{2h-3}^{2h-1} \{a+2)(a+4)\dots(a+2h-4)\} \\ &\quad + 1 \cdot 3 C_{2h-5}^{2h-1} \{(a+4)(a+6)\dots(a+2h-4)\} \\ &\quad + 1 \cdot 3 \cdot 5 C_{2h-7}^{2h-1} \{(a+6)(a+8)\dots(a+2h-4)\} + \dots \\ &\quad + \{1 \cdot 3 \cdot 5 \dots (2h-5)\} C_3^{2h-1} (a+2h-4) + \{1 \cdot 3 \cdot 5 \dots (2h-3)\} C_1^{2h-1} \end{aligned}$$

This identity has been given in Art. 3 of the first part.

$$\begin{aligned} \text{Ex. 4. } \underline{n-1} + \underline{n-2} C_2^{n-k} (a+k-1) + \underline{n-3} C_3^{n-k} (a+k)(a+k-1) \\ + \underline{n-4} C_4^{n-k} (a+k+1)(a+k)(a+k-1) + \dots \\ + \underline{k-1} \{a+n-2)(a+n-3)\dots(a+k-1)\} C_{n-k}^{n-k} \\ = \underline{k-1} \{(a+2k-1)(a+2k)(a+2k+1)\dots(a+n+k-2)\} \end{aligned}$$

identically, where n and k are both positive integers and k is less than n .

This identity may be deduced from the theorem (1) by putting $\delta=1-n, r=n-k$ and $-x=a+n+k-2$.

$$\begin{aligned} \text{Ex. 5. } \frac{\underline{n-h-1}}{\underline{r-1}} C_h^{n-r} C_h^h + \frac{\underline{n-h-2}}{\underline{r-1}} C_{h+1}^{n-r} C_h^{h+1} C_1^{a+r+h-1} \\ + \frac{\underline{n-h-3}}{\underline{r-1}} C_{h+2}^{n-r} C_h^{h+2} C_2^{a+r+h} + \frac{\underline{n-h-4}}{\underline{r-1}} C_{h+3}^{n-r} C_h^{h+3} C_3^{a+r+h+1} C_3 \\ + \dots + \frac{\underline{n-r-h-1}}{\underline{r-1}} C_{n-r-1}^{n-r} C_h^{n-r-1} C_{n-r-h-1}^{a+n-3} \\ + \frac{\underline{n-r-h}}{\underline{r-1}} C_{n-r}^{n-r} C_h^{n-r} C_{n-r-h}^{a+n-2} \\ = C_h \{(a+2r+h-1)(a+2r+h)(a+2r+h+1)\dots(a+n+r-2)\} \end{aligned}$$

identically where h is $\leq n-r$ and $r \leq n$.

Proof. The left-hand-side expression

$$\begin{aligned}
 &= C_k \left[\frac{n-h-1}{r-1} + 1 \frac{n-h-2}{r-1} \frac{n-r-h}{C_1} \frac{a+r+h-1}{C_1} \right. \\
 &+ \frac{2}{r-1} \frac{n-h-3}{C_2} \frac{n-r-h}{C_2} \frac{a+r+h}{C_2} + \frac{3}{r-1} \frac{n-h-4}{C_3} \frac{n-r-h}{C_3} \frac{a+r+h+1}{C_3} \\
 &+ \dots + \frac{r-2}{r-1} \frac{n-r-h-1}{C_{n-r-h-1}} \frac{n-r-h}{C_{n-r-h-1}} \frac{a+n-3}{C_{n-r-h-1}} \\
 &\quad \left. + \frac{n-1}{r-1} \frac{n-r-h}{C_{n-r-h}} \frac{n-r-h}{C_{n-r-h}} \frac{a+n-2}{C_{n-r-h}} \right]
 \end{aligned}$$

If $k=n-h$, $b=a+h$ and hence $b+k=n+a$, then the left-hand-side expression

$$\begin{aligned}
 &= C_k \left[\frac{k-1}{r-1} + \frac{k-2}{r-1} \frac{k-r}{C_1} (b+r-1) + \frac{k-3}{r-1} \frac{k-r}{C_2} (b+r)(b+r-1) \right. \\
 &\quad + \frac{k-4}{r-1} \frac{k-r}{C_3} (b+r+1)(b+r)(b+r-1) + \dots \\
 &\quad \left. + \frac{r-1}{r-1} \frac{k-r}{C_{k-r}} \{(b+k-2)(b+k-3)\dots(b+r-1)\} \right]
 \end{aligned}$$

$$= C_k \frac{n-r}{(b+2r-1)(b+2r)(b+2r+1)\dots(b+k+r-2)} \text{ by example 4.}$$

Hence the left-hand-side expression

$$= C_k \frac{n-r}{(a+h+2r-1)(a+h+2r)(a+h+2r+1)\dots(a+n+r-2)}$$

Thus the identity is proved.

3. If $(a, b, x)_r$ denote the expression $\{a(a+b)(a+2b) \dots (a+r-1b)\}$

$$\begin{aligned} & \dots (a+r-1b)\} - C_1 \{(a-d)(a+b)(a+2b) \dots (a+r-1b)\} x \\ & + C_2 \{(a-d)(a+b-d)(a+2b)(a+3b) \dots (a+r-1b)\} x^2 \\ & - C_3 \{(a-d)(a+b-d)(a+2b-d)(a+3b) \dots (a+r-1b)\} x^3 + \dots \\ & + (-1)^r C_r \{(a-d)(a+b-d) \dots (a+r-1b-d)\} x^r. \end{aligned}$$

then

$$\begin{aligned} (a, b, x)_r - x(a+b, b, x)_r &= \{a(a+b)(a+2b) \dots (a+r-1b)\} \\ &\times (1-x)^{r+1} \text{ identically} \end{aligned} \quad \dots (3)$$

For the coefficient of x^h in $(a, b, x)_r$ is $(-1)^h C_h \{(a-b)a(a+b)(a+2b) \dots (a+h-2b)(a+hb)(a+h+1b) \dots (a+r-1b)\}$ and the coefficient of x^h in $-(a+b, b, x)_r x$ is $(-1)^{h+r} C_{h+r-1} \{a(a+b)(a+2b) \dots (a+h-2b)(a+hb)(a+h+1b) \dots (a+rb)\}$

\therefore the coefficient of x^h in the left-hand-side expression of (3) is $(-1)^h \{a(a+b)(a+2b) \dots (a+h-2b)(a+hb)(a+h+1b) \dots (a+r-1b)\}$.

$$\begin{aligned} & \times [C_h(a-b) + C_{h+r-1}(a+rb)] = (-1)^h \{a(a+b)(a+2b) \\ & \dots (a+h-2b)(a+hb)(a+h+1b) \dots (a+r-1b)\}^{r+1} C_h(a+h-1b) \\ & = (-1)^h \{a(a+b)(a+2b) \dots (a+r-1b)\}^{r+1} C_h. \end{aligned}$$

$$\begin{aligned} \therefore (a, b, x)_r - (a+b, b, x)_r x &= \{a(a+b)(a+2b) \\ & \dots (a+r-1b)\} \{1 - C_1 x + C_2 x^2 - \dots + (-1)^{r+1} C_{r+1} x^{r+1}\} \\ & = \{a(a+b)(a+2b) \dots (a+r-1b)\} (1-x)^{r+1} \end{aligned}$$

$$4. (a+rb)(a, b, x)_r - a(a+b, b, x)_r - rb^2x(a+2b, b, x)_{r-1} = 0$$

identically...

(4)

For, in the left-hand-side expression of (4), the coefficient of x^h is

$$\begin{aligned} & (-1)^r C_h \{ (a-b)a(a+b) \dots (a+\overline{h-2}b)(a+hb) \dots (a+rb) \} \\ & - (-1)^r C_h \{ a^2(a+b)(a+2b) \dots (a+\overline{h-1}b)(a+\overline{h+1}b) \\ & \dots (a+rb) \} + (-1)^{r-1} C_{h-1} \{ a(a+b) \dots (a+\overline{h-2}b)(a+\overline{h+1}b) \dots (a+rb) \} \\ & = (-1)^h \{ a(a+b)(a+2b) \dots (a+\overline{h-2}b)(a+\overline{h+1}b) \dots (a+rb) \} \\ & \quad \times [{}^r C_h (a-b)(a+hb) - {}^r C_h a(a+\overline{h-1}b) + b^2 {}^{r-1} C_{h-1}] \\ & = (-1)^h \{ a(a+b)(a+2b) \dots (a+\overline{h-2}b)(a+\overline{h+1}b) \dots (a+rb) \} \\ & \quad {}^r C_h [(a-b)(a+hb) - a(a+\overline{h-1}b) + hb^2] = 0 \text{ for the expression within} \end{aligned}$$

the brackets [] is zero. Hence the theorem is proved.

5. If S_n denote the sum of the products of the n factors, $1, a, a^2, \dots$

a^{n-1} taken r of them at a time and if $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ denote the product

$$\begin{aligned} & (a^n - 1)(a^{n-1} - 1) \dots (a^r - 1) \text{ then } \left[\begin{smallmatrix} h-1 \\ r \end{smallmatrix} \right] - \frac{1}{a^k} (a^{h-r} - 1) \left[\begin{smallmatrix} h-2 \\ r \end{smallmatrix} \right] S_1^{r+k} \\ & + \frac{1}{a^{2k}} \left[\begin{smallmatrix} h-r \\ h-r-1 \end{smallmatrix} \right] \left[\begin{smallmatrix} h-3 \\ r \end{smallmatrix} \right] S_2^{r+k} - \frac{1}{a^{3k}} \left[\begin{smallmatrix} h-r \\ h-r-2 \end{smallmatrix} \right] \left[\begin{smallmatrix} h-1 \\ r \end{smallmatrix} \right] S_3^{r+k} + \dots \\ & + (-1)^{h-r} \frac{1}{a^{k(h-r)}} \left[\begin{smallmatrix} h-r \\ 1 \end{smallmatrix} \right] S_{h-r}^{r+k} = 0 \text{ where } h, r \text{ and } k \text{ are all positive} \\ & \text{integers } k \text{ varying from } 0 \text{ to } h-r-1.^1 \end{aligned}$$

¹ [Note :- If n is less than r , S_r is to be taken as zero but $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ as unity. If

$n=r$ then $\left[\begin{smallmatrix} n \\ r \end{smallmatrix} \right]$ denotes a single factor viz., $a^n - 1$.]

(i) Let us take the series $u_1, u_2, u_3, u_4, \dots$ and obtain from it another series by subtracting each term from the term which immediately precedes it. The series $u_1 - S_1 u_2, u_2 - S_1 u_3, u_3 - S_1 u_4, \dots$, thus found, may be called the series of the first order of differences and let this series be denoted by Δ_1 . Multiply each term, of Δ_1 , by u and subtract the product from the term which immediately precedes it, then we get the series of the second order of differences *viz.*,

$u_1 - S_1 u_2 + S_2 u_3, u_2 - S_1 u_3 + S_2 u_4, u_3 - S_1 u_4 + S_2 u_5, \dots$ which may be denoted by Δ_2 .

Similarly we are to get Δ_3 from Δ_2 by using u^2 as a multiplier
 $\dots \Delta_4 \dots \quad \Delta_5 \dots \quad u^3 \dots$

and so on.

Here we observe that some formula holds in the case of each term of any of the series Δ_1 and Δ_2 . Let us assume that this formula holds in the case of Δ_{r-1} *i.e.*, suppose Δ_{r-1} is

$$u_1 - S_1 u_2 + S_2 u_3 - \dots + (-1)^{r-1} S_{r-1} u_r, u_2 - S_1 u_3 + \dots \\ + \dots (-1)^{r-1} S_{r-1} u_{r+1}, \dots$$

Then by multiplying each term, by u^{r-1} and subtracting the product from the term that immediately precedes it, we get the r^{th} order of differences *viz.*,

$$u_1 - S_1 u_2 + S_2 u_3 - \dots (-1)^{r-1} S_{r-1} u_{r+1}, u_2 - S_1 u_3 + S_2 u_4 - \dots \\ - \dots (-1)^{r-1} S_{r-1} u_{r+1}, \dots \left[\text{for } S_m + u^{r-1} (-1)^{r-1} S_{m-1} = S_m^r \right].$$

Thus if the formula holds in the case of Δ_{r-1} , it also holds in the case of Δ_r . But it holds in the case of Δ_1 , and Δ_2 and hence it holds universally,

(ii). If A_r^n denote the sum of the products of n factors a, a^2, a^3, \dots, a^n taken r of them at a time then it is evident that $r A_r + {}^{r-1} A_{r-1} = {}^r A_r + 1 \cdot {}^{r-1} A_{r-1} = {}^{r+1} S_1$ and since every term of A_r^h is the product of r factors and each of these factors, multiplied by a , gives the

corresponding factor in the corresponding term of A_r , we have

$$a^r S_r = A_r.$$

(iii). Let us take the series Δ_r viz

$$u_1 - {}^r S_1 u_2 + {}^r S_2 u_3 - \dots + (-1)^r {}^r S_{r+1} u_{r+2} - {}^r S_1 u_3 + {}^r S_2 u_4 \\ - \dots + (-1)^r {}^r S_{r+1} u_{r+3}, \dots$$

Multiply each term, of Δ_r by $\frac{1}{a}$ and subtract the product from the term which immediately precedes it. Let the series, thus found, be denoted by D_1 . Then the first term of D_1 is

$$u_1 - \frac{1}{a} (a {}^r S_1 + 1) u_2 + \frac{1}{a^2} (a^2 {}^r S_2 + a {}^r S_1) u_3 - \dots \\ + (-1)^r \frac{1}{a^r} (a^r {}^r S_r + a^{r-1} {}^r S_{r-1}) u_{r+1} + (-1)^{r+1} \frac{1}{a^{r+1}} a^r {}^r S_{r+1} u_{r+2} \\ = u_1 - \frac{1}{a} (A_1 + 1) u_2 + \frac{1}{a^2} (A_2 + A_1) u_3 \dots \\ + (-1)^r \frac{1}{a^r} (A_r + A_{r-1}) u_{r+1} + (-1)^{r+1} \frac{1}{a^{r+1}} A_{r+1} u_{r+2} \text{ by (ii)} \\ = u_1 - \frac{1}{a} {}^r S_1 u_2 + \frac{1}{a^2} {}^r S_2 u_3 \dots \\ + (-1)^r \frac{1}{a^r} {}^r S_r u_{r+1} + (-1)^{r+1} \frac{1}{a^{r+1}} {}^r S_{r+1} u_{r+2} \text{ by (ii)}$$

Similarly the other terms of D_1 may be obtained.

We can similarly show that if the series obtained from D_1 , by using $\frac{1}{a^2}$ as a multiplier, be denoted by D_2 , then D_2 is

$$u_1 - \frac{1}{a^2} {}^{r+2} S_1 u_2 + \frac{1}{a^2} {}^{r+2} S_2 u_3 - \dots + (-1)^{r+2} \frac{1}{a^{r+2}} {}^{r+2} S_{r+2} u_{r+3} - \frac{1}{a^2} {}^{r+2} S_1 u_3 \\ + \dots + (-1)^{r+2} \frac{1}{a^{r+2}} {}^{r+2} S_{r+2} u_{r+3}, \dots$$

Thus D_3, D_4, \dots may be obtained by using $\frac{1}{a^3}, \frac{1}{a^4}, \dots$ as multipliers.

Then by the method of induction we can show that D_k is the series

$$u_1 - \frac{1}{a^k} S_1 u_2 + \frac{1}{a^{2k}} S_2 u_3 - \dots + (-1)^{r+k} \frac{1}{a^{k(r+k)}} S_{r+k} u_{r+k+1},$$

$$u_2 - \frac{1}{a^k} S_1 u_3 + \dots + (-1)^{r+k} \frac{1}{a^{k(r+k)}} S_{r+k} u_{r+k+2}, \dots$$

(iv). Let us now take the series

$$\left[\begin{matrix} h-1 \\ r \end{matrix} \right], \left(a^{h-r} - 1 \right) \left[\begin{matrix} h-2 \\ r \end{matrix} \right], \left[\begin{matrix} h-r \\ h-r-1 \end{matrix} \right] \left[\begin{matrix} h-3 \\ r \end{matrix} \right],$$

$$\left[\begin{matrix} h-r \\ h-r-2 \end{matrix} \right] \left[\begin{matrix} h-4 \\ r \end{matrix} \right], \dots, \left[\begin{matrix} h-r \\ 2 \end{matrix} \right] (a^r - 1)$$

$$\left[\begin{matrix} h-r \\ 1 \end{matrix} \right], 0, 0, 0, 0 \dots$$

Then

$$\Delta_1 \text{ is } a^{h-r} \left[\begin{matrix} h-2 \\ r-2 \end{matrix} \right], a^{h-r-1} \left(a^{h-r} - 1 \right) \left[\begin{matrix} h-3 \\ r-1 \end{matrix} \right],$$

$$a^{h-r-2} \left[\begin{matrix} h-r \\ h-r-1 \end{matrix} \right] \left[\begin{matrix} h-4 \\ r-1 \end{matrix} \right] \dots a \left[\begin{matrix} h-r \\ 2 \end{matrix} \right] (a^{r-1} - 1),$$

$$\left[\begin{matrix} h-r \\ 1 \end{matrix} \right], 0, 0, 0 \dots$$

$$\Delta_2 \text{ is } a^{2(h-r)} \left[\begin{matrix} h-3 \\ r-2 \end{matrix} \right], a^{2(h-r-1)} \left(a^{h-r} - 1 \right) \left[\begin{matrix} h-4 \\ r-2 \end{matrix} \right],$$

$$a^{2(h-r-2)} \left[\begin{matrix} h-r \\ h-r-1 \end{matrix} \right] \left[\begin{matrix} h-5 \\ r-2 \end{matrix} \right] \dots a^2 \left[\begin{matrix} h-r \\ 2 \end{matrix} \right] (a^{r-2} - 1),$$

$$\left[\begin{matrix} h-r \\ 1 \end{matrix} \right], 0, 0, 0 \dots$$

...

$$\Delta_{r-1} \text{ is } a^{(r-1)(h-r)} \left[\begin{matrix} h-r \\ 1 \end{matrix} \right], a^{(r-1)(h-r-1)} \left[\begin{matrix} h-r \\ 1 \end{matrix} \right],$$

$$a^{(r-1)(h-r-2)} \left[\begin{matrix} h-r \\ 1 \end{matrix} \right], \dots a^{r-1} \left[\begin{matrix} h-r \\ 1 \end{matrix} \right] \left[\begin{matrix} h-r \\ 1 \end{matrix} \right], 0, 0, 0 \dots$$

$$\Delta_r \text{ is } 0, 0, 0 \dots, 0, \left[\begin{matrix} h-r \\ 1 \end{matrix} \right], 0, 0, 0 \dots$$

Hence all the terms except $h-r+1^{th}$ of Δ_r , except $h-r+1^{th}$ and $h-r^{th}$ of D_1 , except $h-r+1^{th}$, $h-r^{th}$ and $h-r-1^{th}$ of D_2 etc are zero. In the case of D_{h-r-1} , the first term is zero but all the terms from the 2^{nd} to the $h-r+1^{th}$ do not vanish. Thus the first term of each of $\Delta_r, D_1, D_2, \dots, D_{h-r-1}$ is zero. Hence by (i) and (iii) we have

$$\begin{aligned} & \left[\begin{matrix} h-1 \\ r \end{matrix} \right] - \frac{1}{a^k} \left(a^{h-r} - 1 \right) \left[\begin{matrix} h-2 \\ r \end{matrix} \right] S_1^{r+k} + \frac{1}{a^{2k}} \left[\begin{matrix} h-r \\ h-r-1 \end{matrix} \right] \left[\begin{matrix} h-3 \\ r \end{matrix} \right] S_2^{r+k} \\ & - \dots + (-1)^{h-r} \frac{1}{a^{k(h-r)}} \left[\begin{matrix} h-r \\ 1 \end{matrix} \right] S_{h-r}^{r+k} = 0 \end{aligned}$$

where k varies from 0 to $h-r-1$.

It is interesting to note in this connection that if S_r denotes the sum of the products of n factors $a_1, a_2, a_3 \dots a_n$ taken r of them at a time where a 's are all arbitrary, then using a_1, a_2, a_3 etc. as successive multipliers it can easily be shown by induction that the r^{th} order of differences is the series

$$\begin{aligned} & u_1 - S_1^r u_2 + S_2^r u_3 - \dots (-1)^r S_r^r u_{r+1}, u_2 - S_1^r u_3 + S_2^r u_4 \\ & - \dots (-1)^r S_r^r u_{r+2}, \dots \left[\text{for } S_k + a_{r+1} S_{k-1} = S_k^{r+1} \right] \end{aligned}$$

$$6. \quad S_k^{k+r} = \frac{\left[\begin{matrix} k+r \\ 1 \end{matrix} \right]}{\left[\begin{matrix} r \\ 1 \end{matrix} \right] \left[\begin{matrix} k \\ 1 \end{matrix} \right]} S_k \text{ identically.}$$

$$h \quad h+1 \quad h+2$$

Let T_k denote the series $S_k, S_{k+1}, S_{k+2}, \dots$ then we find by trial that the theorem holds good in the case of each term of the series T_1 . Let us assume that it holds in the case of the series T_{k-1} .

The first term of T_k is S_k .

$$\begin{aligned} & \text{The second term of } T_k \text{ is } S_k^{k+1} = S_k + a^k S_{k-1} \\ & = S_k + a^k \frac{a^k - 1}{a - 1} S_{k-1} = S_k \left(1 + a \frac{a^k - 1}{a - 1} \right) = \frac{a^{k+1} - 1}{a - 1} S_k \end{aligned}$$

$$\frac{\left[\begin{matrix} k+1 \\ 1 \end{matrix} \right]}{\left[\begin{matrix} k \\ 1 \end{matrix} \right] \left[\begin{matrix} 1 \\ 1 \end{matrix} \right]} S_k$$

Thus we see that the theorem holds in the case of the first two terms of T_k . Suppose it holds good in the case of ${}^{k+r-1}S_k$.

$$\begin{aligned}
 \text{But } {}^{k+r}S_k &= {}^{k+r-1}S_k + a^{k+r-1} {}^{k+r-1}S_{k-1} = \frac{\left[\begin{smallmatrix} k+r-1 \\ k+1 \end{smallmatrix} \right]}{\left[\begin{smallmatrix} r-1 \\ 1 \end{smallmatrix} \right]} {}^kS_k \\
 &\quad + a^{k+r-1} \frac{\left[\begin{smallmatrix} k+r-1 \\ k \end{smallmatrix} \right]}{\left[\begin{smallmatrix} r \\ 1 \end{smallmatrix} \right]} {}^{k-1}S_{k-1} = \frac{\left[\begin{smallmatrix} k+r-1 \\ k+1 \end{smallmatrix} \right]}{\left[\begin{smallmatrix} r-1 \\ 1 \end{smallmatrix} \right]} \\
 &\times \left\{ {}^kS_k + a^{k+r-1} \frac{a^k-1}{a^r-1} {}^{k-1}S_{k-1} \right\} = \frac{\left[\begin{smallmatrix} k+r-1 \\ k+1 \end{smallmatrix} \right]}{\left[\begin{smallmatrix} r-1 \\ 1 \end{smallmatrix} \right]} {}^kS_k \left\{ 1 + a^r \frac{a^k-1}{a^r-1} \right\} \\
 &= \frac{\left[\begin{smallmatrix} k+r \\ k+1 \end{smallmatrix} \right]}{\left[\begin{smallmatrix} r \\ 1 \end{smallmatrix} \right]} {}^kS_k = \frac{\left[\begin{smallmatrix} k+r \\ 1 \end{smallmatrix} \right]}{\left[\begin{smallmatrix} k \\ 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} r \\ 1 \end{smallmatrix} \right]}
 \end{aligned}$$

Hence if the theorem is true for $r-1$ 'th term of the series T_k , it is also true for the r 'th term of the same series. But it is true for the first two terms of the series T_k . So it holds in the case of each term of the series T_k . Thus we see that if it holds in the case of the series T_{k-1} , it also holds in the case of the series T_k . But it holds in the case of T_1 . Hence it holds universally.

$$\begin{aligned}
 7. \quad & 1 - \frac{a^r(a^{n-k-r}-1)(1+a^{r+k-1}y)}{(a-1)(a^{n-k-1}-1)} \\
 & + \frac{a^{2r} \left[\begin{smallmatrix} n-k-r \\ n-k-r-1 \end{smallmatrix} \right] (1+a^{r+k-1}y)(1+a^{r+k}y)}{\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} n-k-1 \\ n-k-2 \end{smallmatrix} \right]} \\
 & - \frac{a^{3r} \left[\begin{smallmatrix} n-k-r \\ n-k-r-2 \end{smallmatrix} \right] \{ (1+a^{r+k-1}y)(1+a^{r+k}y)(1+a^{r+k+1}y) \}}{\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} n-k-1 \\ n-k-3 \end{smallmatrix} \right]} \\
 & + \dots + (-1)^{n-k-r} \frac{(n-k-r)r}{a}
 \end{aligned}$$

$$\begin{aligned}
& \times \frac{\left[\begin{smallmatrix} n-k-r \\ 1 \end{smallmatrix} \right] \left\{ \left(1+a^{r+k-1}y \right) \left(1+a^{r+k}y \right) \cdots \left(1+a^{n-2}y \right) \right\}}{\left[\begin{smallmatrix} n-k-r \\ 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} n-k-1 \\ r \end{smallmatrix} \right]} \\
& = (-1)^{n-k-r} \frac{\left\{ \left(1+a^{r+k-1}y \right) \left(1+a^{r+k}y \right) \cdots \left(1+a^{n-2}y \right) \right\}}{\left[\begin{smallmatrix} n-k-1 \\ r \end{smallmatrix} \right]}
\end{aligned}$$

identically.

Proof. If $y = -\frac{1}{a^{r+k-1}}$, each of the two expressions of the above is equal to unity.

If $y = -\frac{1}{a^{2r+k-1}}$, the right-hand-side expression = 0 and the left-hand-side expression

$$\begin{aligned}
& = 1 - \frac{\left(a^{n-k-r} - 1 \right) \left(\begin{smallmatrix} r \\ a-1 \end{smallmatrix} \right)}{(a-1) \left(\begin{smallmatrix} n-k-1 \\ a-1 \end{smallmatrix} \right)} + \frac{a \left[\begin{smallmatrix} n-k-r \\ n-k-r-1 \end{smallmatrix} \right] \left[\begin{smallmatrix} r \\ r-1 \end{smallmatrix} \right]}{\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} n-k-1 \\ n-k-2 \end{smallmatrix} \right]} \\
& \quad + \frac{a^2 \left[\begin{smallmatrix} n-k-r \\ n-k-r-2 \end{smallmatrix} \right] \left[\begin{smallmatrix} r \\ r-2 \end{smallmatrix} \right]}{\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} n-k-1 \\ n-k-3 \end{smallmatrix} \right]} + \dots \\
& + (-1)^{n-k-r} \frac{a^{\frac{1}{2}(n-k-r)(n-k-r-1)} \left[\begin{smallmatrix} n-k-r \\ 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} 2r+k-n+1 \\ r \end{smallmatrix} \right]}{\left[\begin{smallmatrix} n-k-r \\ 1 \end{smallmatrix} \right] \left[\begin{smallmatrix} n-k-1 \\ r \end{smallmatrix} \right]} \\
& = 1 - \frac{a^{n-k-r} - 1}{a^{n-k-1} - 1} {}^r S_1 + \frac{\left[\begin{smallmatrix} n-k-r \\ n-k-r-1 \end{smallmatrix} \right]}{\left[\begin{smallmatrix} n-k-1 \\ n-k-2 \end{smallmatrix} \right]} {}^r S_2 \dots \dots \\
& \quad + (-1)^{n-k-r} \frac{\left[\begin{smallmatrix} n-k-r \\ 1 \end{smallmatrix} \right]_r}{\left[\begin{smallmatrix} n-k-1 \\ r \end{smallmatrix} \right]} S_{n-k-r}
\end{aligned}$$

by Art 6

$$\begin{aligned}
&= \frac{1}{\left[\begin{smallmatrix} n-k-1 \\ r \end{smallmatrix} \right]} \left\{ \left[\begin{smallmatrix} n-k-1 \\ r \end{smallmatrix} \right]^r S_{n-k-r} \right. \\
&- \left[\begin{smallmatrix} n-k-2 \\ r \end{smallmatrix} \right] (a^{n-k-r}-1)^r S_1 + \left[\begin{smallmatrix} n-k-3 \\ r \end{smallmatrix} \right] \left[\begin{smallmatrix} n-k-r \\ n-k-r-1 \end{smallmatrix} \right]^r S_1 \\
&- \dots + (-1)^{n-k-r} \left[\begin{smallmatrix} n-k-r \\ 1 \end{smallmatrix} \right]^r S_{n-k-r} \left. \right\} = 0
\end{aligned}$$

for, if we put $n-k=h$, the expression within the brackets becomes

$$\begin{aligned}
&\left[\begin{smallmatrix} h-1 \\ r \end{smallmatrix} \right] - (a^{h-r}-1)^r \left[\begin{smallmatrix} h-2 \\ r \end{smallmatrix} \right]^r S_1 + \left[\begin{smallmatrix} h-r \\ h-r-1 \end{smallmatrix} \right] \left[\begin{smallmatrix} h-3 \\ r \end{smallmatrix} \right]^r S_1 - \dots \\
&+ (-1)^{h-r} \left[\begin{smallmatrix} h-r \\ 1 \end{smallmatrix} \right]^r S_{h-r}
\end{aligned}$$

which is zero by Art 5.

Similarly by means of the theorem given in Art 5, we can show that both the expressions of the equation are zero, when we substitute any

of the values $-\frac{1}{a^{2r+k}}, -\frac{1}{a^{2r+k+1}}, \dots, -\frac{1}{a^{r-2}}$ for y in the equation

Thus for $n-k-r+1$ values of y the equation is satisfied. Hence it is an identity.

As source of references is too difficult to be available here, so if any of the above results have been discovered by other mathematicians, we shall be very glad to mention their names in proper places.

ON THE PERTURBATIONS OF THE ORBIT OF THE VALENCY-ELECTRON IN THE GENERALIZED HYDROGEN-UNLIKE ATOM (A)

By
K. BASU.

INTRODUCTION.

According to the modern theories of atomic structure, the atom consists of a nuclear positive charge Ne , with N electrons rotating round it in different successive shells (N =atomic number). Of late years, attempt has been made to explain the spectral lines as well as the chemical properties of the atom on a dynamical quantum theory of the orbital motion of the electrons. The first attempt in this direction was made by Bohr¹; by combining the quantum theory of energy exchanges with the nuclear theory of the atom, Bohr was able to explain very successfully the spectral series of hydrogen and ionized helium (He^+).

Bohr's method was generalized by Sommerfeld² in a remarkable series of papers. With the aid of the generalized theory of quantum vibration, Sommerfeld succeeded in explaining in a qualitative manner the spectral series of alkalis and ionized alkaline earths, and in laying down certain general rules for the elucidation of the spectra of elements. Further progress in this direction is hampered by our inability to cope with the time-honoured problem of three bodies.

The problem is to find out the motion of any one of the electrons in the combined field of the nucleus and the other electrons according to quantum-mechanics. When the electron happens to be the outermost valency electron, the solution of its motion would provide us with the key to the explanation of its visible spectra. If it happens to be any

¹ Phil. Mag. July, 1913 *et. seqq.*

² Sommerfeld "*Atombau und Spektrallinien.*" Chap. 4.

one of the inner electrons the solution would enable us to explain the K—, L—, M— radiations in the X-ray region.¹

Since an exact solution is not yet in sight, attempts have been made to obtain approximate solutions. Thus Lande² and Bohr³ have tried to tackle helium ($N=2$), Sommerfeld⁴ tried to tackle the general case of motion of the outer electron, assuming the total charge of the electrons to be equally distributed in a ring of radius a . But as we know from other sources of evidence, this is far removed from the actual state of affairs. The electrons are arranged in different shells, containing 2, 8, 8, 18, 18, 32, ... electrons which move according to definite quantum-conditions.⁵ The problem is therefore to find out (1) The electrical field due to electrons moving in definite shells about the nucleus, (2) to investigate the motion of the outer electron in the combined field.

In the following I have assumed that n electrons, situated at the corners of a regular polygen of n -sides are rotating with angular velocity ω about the nucleus. The general field due to such a ring being found, we can obtain the total field by simple addition. The range of validity of Sommerfeld's assumption has also been investigated. And as a matter of fact, it is very probable that the outer electron cannot describe the same circular orbit permanently under the action of n rapidly moving electrons, on the contrary, it may suffer periodical perturbation, the present attempt aims at determining the perturbed orbit of such an atom conventionally known as hydrogen-unlike (*Wasserstoffunähnlich*).

The ring configuration having $Z-k$ electrons (Z =atomic number) was tackled by Sommerfeld⁶ on an assumption of sufficient quickness (*hinreichend rasch*) of revolution of the electrons and he calculated the energy function by a method of approximations, in terms of quantum numbers, which can be utilized to frame the *Haupt*, *Neben* and *Bergmann* series formulae. The type of such formulae is quite different from that of the hydrogen atom, so the name "hydrogen-unlike atoms" is prescribed to signify another type of series formulae. Such ring

¹ Kossel, "Zs. f. Physik," Vol. 2, p. 470; Wentzel, "Zs. f. Physik." Vol. 8, p. 85; Coster, "Phil. Mag.," 1922.

² Lande, "Phys. Zs.," 1921, p. 114.

³ Bohr, "Zs. f. Physik.," Vol. 9, pp. 1-67.

⁴ Sommerfeld, "Atombau, third ed.," Anhang, p. 721.

⁵ Loring, "Atomic Theories."

⁶ 'Atombau und Spektrallinien' Zweite Auflage, Braunschweig, 1921, *Zusätze und Ergänzungen*, § 10 pp. 506-14.

configurations have been applied provisionally with apparent success to a wide range of phenomena, notably, the theoretical derivation of Ritz formula and *a fortiori* of the Balmer and Rydberg ones, and the computation of X-ray frequencies.

• The additional field (*Zusatfeld*) of Sommerfeld is modified in this paper to fit in with the assumption that the angular velocity of the ring electrons will not be indefinitely large in comparison with that of the valency electron, provided that under certain legitimate limitations the ring configuration as postulated is a stable one¹, and the ionization potentials found out theoretically on such a basis is in agreement with experimental facts.

In fact the term *generalized* is appended to signify an electrically neutral atom, although the present problem is applicable quite well to ionized atoms, and without loss of generality to all heavy atoms ionized to have a single valency electron, since the ring next to the latter has the most important bearing on its motion than other interior rings. Although, as a matter of fact, the theory of ring configuration of atomic systems is losing much of its interests and Bohr (*loc-cit*) conceives of separate orbital configurations for each electron still it is quite apropos of the time which has hardly ever any permanency of theoretical grounds.

II

DETERMINATION OF POTENTIAL FUNCTION.

(a) *By the Method of Zonal Harmonics.*

Suppose generally there are n electrons situated at the corners of a regular n -gon, which is rotating with the velocity ω . We wish to find out the potential at an external point (r, θ) having a charge $-e$, the initial line passing through the centre and a particular electron on the ring.

We have

$$V = -\frac{Ze^2}{r} + e^2 \left(\frac{1}{r_1} + \frac{1}{r_1} + \dots + \frac{1}{r_n} \right)$$

where r_1, r_2, \dots, r_n are the distances of the electrons from (r, θ) , the *Aufpunkt*, and $+Ze$ the central charge.

The stability will be discussed in paper B in the next issue of this *Bulletin*.

Now

$$r_1^2 = r^2 + a^2 - 2racos\theta$$

$$r_2^2 = r^2 + a^2 - 2racos(\theta + a), \text{ etc.,}$$

$$r_s^2 = r^2 + a^2 - 2racos(\theta + s-1a), \text{ where } a = \frac{2\pi}{n}.$$

We have

$$\frac{1}{r_s} = \frac{1}{r} \left(1 + \frac{a}{r} P_1(\theta + s-1a) + \dots + \left(\frac{a}{r} \right)^m P_m(\theta + s-1a) + \dots \right).$$

For brevity, let us denote

$$P_m(\theta + s-1a) \text{ by } P_{m,s-1}.$$

Then

$$\begin{aligned} V = & -\frac{Ze^2}{r} + \frac{e^2}{r} \left[1 + \left(\frac{a}{r} \right) P_{1,0} + \left(\frac{a}{r} \right)^2 P_{2,0} + \dots \right] \\ & + \frac{e^2}{r} \left[1 + \left(\frac{a}{r} \right) P_{1,1} + \left(\frac{a}{r} \right)^2 P_{2,1} + \dots \right] + \text{etc.,} \\ & + \frac{e^2}{r} \left[1 + \left(\frac{a}{r} \right) P_{1,n-1} + \left(\frac{a}{r} \right)^2 P_{2,n-1} + \dots \right]. \end{aligned}$$

That is

$$\begin{aligned} V = & -\frac{Ze^2}{r} + \frac{e^2}{r} \left[n + \left(\frac{a}{r} \right) S_1^n + \left(\frac{a}{r} \right)^2 S_2^n + \dots \right], \text{ if} \\ S_m^n = & P_{m,0} + P_{m,1} + P_{m,2} + \dots + P_{m,n-1}. \end{aligned}$$

We know

$$\begin{aligned} P_{m,0} = & \frac{1 \cdot 3 \cdot 5 \dots 2m-1}{2 \cdot 4 \cdot 6 \dots 2m} \left[2cosm\theta + 2 \frac{1 \cdot m}{1 \cdot 2m-1} cos(m-2)\theta \right. \\ & \left. + 2 \cdot \frac{1 \cdot 3 \cdot m(m-1)}{1 \cdot 2 \cdot (2m-1)(2m-3)} cos(m-4)\theta + \dots \right], \end{aligned}$$

similar values for $P_{m,1}$, $P_{m,2}$, etc. [See Byerly's Spherical Harmonics p. 159.]

Whence

$$\begin{aligned}
 S_m^* = & \frac{1 \cdot 3 \cdot 5 \dots 2m-1}{2 \cdot 4 \cdot 6 \dots 2m} [2\{\cos m\theta + \cos m(\theta+a) + \cos m(\theta+2a) \\
 & + \dots + \cos m(\theta+n-1a)\} + 2 \cdot \frac{1 \cdot m}{1 \cdot 2m-1} \{\cos(m-2)\theta \\
 & + \cos(m-2)(\theta+a) + \dots + \cos(m-2)(\theta+n-1a)\} \\
 & + 2 \cdot \frac{1 \cdot 3 \cdot m(m-1)}{1 \cdot 2(2m-1)(2m-3)} \{\cos(m-4)\theta + \cos(m-4)(\theta+a) \\
 & + \dots + \cos(m-4)(\theta+n-1a)\} + \dots].
 \end{aligned}$$

Put

$$\begin{aligned}
 C_s^*(\theta) = & \cos s\theta + \cos s(\theta+a) + \dots + \cos s(\theta+n-1a) \cdot \\
 & (s=m, m-2, m-4, \text{etc.})
 \end{aligned}$$

$$\begin{aligned}
 \therefore C_s^*(\theta) = & \cos \left\{ s\theta + (n-1) \frac{sa}{2} \right\} \sin \frac{n}{2} sa \cdot \sin \frac{sa}{2} \\
 = & \cos \left\{ s\theta + \frac{n-1}{n} s\pi \right\} \sin s\pi / \sin \frac{s\pi}{n} \\
 = & 0, (s \neq n).
 \end{aligned}$$

$$C_n^*(\theta) = \cos \{n\theta + (n-1)\pi\} \sin n\pi / \sin \pi.$$

Now, if n odd, $\sin n\pi / \sin \pi = n$; if n even, $\sin n\pi / \sin \pi = -n$; therefore $C_n^*(\theta) = \pm n \cos \{n\theta + (n-1)\pi\}$, according as n is odd or even $= n \cos n\theta$, whether n be even or odd; and $C_{2n}^*(\theta) = \cos \{2n\theta + (n-1)\pi\} \sin 2n\pi / \sin 2\pi = n \cos 2n\theta$, whether n be even or odd; and so on for $C_{r,n}^*(\theta)$, ($r=1, 2, \dots, \text{ad inf.}$).

When m even,

$$S_m^* = n \left\{ \frac{1 \cdot 3 \cdot 5 \dots m-1}{2 \cdot 4 \cdot 6 \dots m} \right\}^{\frac{1}{2}}; \text{ (Byerly, loc. cit.)}$$

when m odd,

$$S_m^* = 0, (m \neq n).$$

And
$$S_n^* = \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} \cdot 2C_n^*(\theta)$$

$$= \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} \cdot 2n \cos n\theta;$$

$$S_{n+2}^* = \frac{1 \cdot 3 \cdot 5 \dots 2n+3}{2 \cdot 4 \cdot 6 \dots 2n+4} \left[2C_{n+2}^*(\theta) + 2 \cdot \frac{1 \cdot n+2}{1 \cdot 2n+3} C_n^*(\theta) \right]$$

$$= \frac{1 \cdot 3 \cdot 5 \dots 2n+3}{2 \cdot 4 \cdot 6 \dots 2n+4} \cdot 2 \cdot \frac{1 \cdot n+2}{1 \cdot 2n+3} n \cos n\theta,$$

all others, except $C_n^*(\theta)$ vanish

$$(\because C_m^*(\theta) = 0, m \neq n \text{ as shown ante}).$$

$$S_{n+1}^* = \frac{1 \cdot 3 \cdot 5 \dots 2n+7}{2 \cdot 4 \cdot 6 \dots 2n+8} [2C_{n+1}^*(\theta) + 2 \cdot \frac{1 \cdot n+4}{1 \cdot 2n+7} C_{n+2}^*(\theta)$$

$$+ 2 \cdot \frac{1 \cdot 3}{1 \cdot 2} \cdot \frac{(n+4)(n+3)}{(2n+7)(2n+5)} \cdot C_n^*(\theta) + \dots]$$

$$= \frac{1 \cdot 3 \cdot 5 \dots 2n+7}{2 \cdot 4 \cdot 6 \dots 2n+8} \cdot 2 \cdot \frac{1 \cdot 3}{1 \cdot 2} \cdot \frac{(n+4)(n+3)}{(2n+7)(2n+5)} n \cos n\theta;$$

all other terms contributing nothing, excepting $C_n^*(\theta)$; and so on.

Hence

$$\begin{aligned} V = & -\frac{Ze^3}{r} + \frac{ne^3}{r} \left[1 + \left(\frac{1}{2} \right)^3 \left(\frac{a}{r} \right)^3 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^3 \left(\frac{a}{r} \right)^4 \right. \\ & \left. + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^3 \left(\frac{a}{r} \right)^5 + \dots \right] \\ & + \frac{e^3}{r} \left[\left\{ \left(\frac{a}{r} \right)^n 2 \cdot \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} n \cos n\theta + \left(\frac{a}{r} \right)^{n+2} 2 \cdot \frac{1 \cdot 3 \cdot 5 \dots 2n+3}{2 \cdot 4 \cdot 6 \dots 2n+4} \right. \right. \\ & \left. \frac{1 \cdot n+2}{1 \cdot 2n+3} n \cos n\theta + \left(\frac{a}{r} \right)^{n+4} 2 \cdot \frac{1 \cdot 3 \cdot 5 \dots 2n+7}{2 \cdot 4 \cdot 6 \dots 2n+8} \cdot \frac{1 \cdot 3(n+4)(n+3)}{1 \cdot 2(2n+7)(2n+5)} \right. \\ & \left. \left. n \cos n\theta + \text{etc.} \right\} + \left\{ \left(\frac{a}{r} \right)^{n-2} 2 \cdot \frac{1 \cdot 3 \cdot 5 \dots 4n-1}{2 \cdot 4 \cdot 6 \dots 4n} n \cos 2n\theta \right. \right. \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{a}{r} \right)^{2n+2} 2 \cdot \frac{1 \cdot 3 \cdot 5 \dots 4n+3}{2 \cdot 4 \cdot 6 \dots 4n+4} \cdot \frac{1 \cdot 2n+2}{1 \cdot 4n+3} n \cos 2n\theta + \left(\frac{a}{r} \right)^{2n+4} \\
 & \cdot \left. \begin{aligned} & 2 \cdot \frac{1 \cdot 3 \cdot 5 \dots 4n+7}{2 \cdot 4 \cdot 6 \dots 4n+8} \cdot \frac{1 \cdot 3}{1 \cdot 2} \cdot \frac{(2n+4)(2n+3)}{(4n+7)(4n+5)} n \cos 2n\theta + \text{etc.} \end{aligned} \right\} + \text{etc.} \Big]. \\
 & = -\frac{Ze^2}{r} + \frac{ne^2}{r} \left[1 + \left(\frac{1}{2} \right)^2 \left(\frac{a}{r} \right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \left(\frac{a}{r} \right)^4 + \dots \right] \\
 & \quad + \frac{2ne^2}{r} [f_n(r) \cos n\theta + f_{2n}(r) \cos 2n\theta + \text{etc.}],
 \end{aligned}$$

where

$$\begin{aligned}
 f_n(r) &= a_n \left(\frac{a}{r} \right)^n + \beta_n \left(\frac{a}{r} \right)^{n+2} + \gamma_n \left(\frac{a}{r} \right)^{n+4} + \dots, \\
 f_{2n}(r) &= a_{2n} \left(\frac{a}{r} \right)^{2n} + \beta_{2n} \left(\frac{a}{r} \right)^{2n+2} + \gamma_{2n} \left(\frac{a}{r} \right)^{2n+4} + \text{etc.},
 \end{aligned}$$

and so on ;

$$\begin{aligned}
 a_n &= \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} ; \beta_n = \frac{1}{2} \cdot \frac{2n+1}{2n+2} a_n ; \\
 \gamma_n &= \frac{1 \cdot 3}{8} \cdot \frac{2n+1}{2n+2} \cdot \frac{2n+3}{2n+4} a_n = \frac{3}{4} \cdot \frac{2n+3}{2n+4} \beta_n ; \text{ etc.}
 \end{aligned}$$

$f_{2n}(r)$, $f_{3n}(r)$, etc., being the same functions of r as $f_n(r)$ obtained by simply substituting $2n$, $3n$, etc., instead of n in $f_n(r)$.

(b) *By the Method of Fourier Series.*

As in (a),

$$\begin{aligned}
 V &= -\frac{Ze^2}{r} + \frac{e^2}{r} [(1-2\mu \cos \theta + \mu^2)^{-\frac{1}{2}} + (1-2\mu \cos \theta + \mu^2)^{-\frac{1}{2}} \\
 & \quad + (1-2\mu \cos(\theta+2\alpha) + \mu^2)^{-\frac{1}{2}} + \dots \\
 & \quad + \{1-2\mu \cos(\theta+n-1\alpha) + \mu^2\}^{-\frac{1}{2}}]
 \end{aligned}$$

when

$$\mu = \frac{a}{r} < 1.$$

Hence

$$V = -\frac{Ze^2}{r} + \frac{e^2}{r} \left[\frac{1}{2} \sum_{-\infty}^{\infty} b^{(i)} \cos i\theta + \frac{1}{2} \sum_{-\infty}^{\infty} b^{(i)} \cos i(\theta + \alpha) \right. \\ \left. + \frac{1}{2} \sum_{-\infty}^{\infty} b^{(i)} \cos i(\theta + 2\alpha) + \dots + \frac{1}{2} \sum_{-\infty}^{\infty} b^{(i)} \cos i(\theta + n-1\alpha) \right],$$

where

$$\frac{1}{2} b^{(i)} = \frac{1 \cdot 3 \cdot 5 \dots 2i-1}{2 \cdot 4 \cdot 6 \dots 2i} \mu^i \left[1 + \frac{1}{2} \cdot \frac{2i+1}{2i+2} \mu^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{(2i+1)(2i+3)}{(2i+2)(2i+4)} \right. \\ \left. \mu^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{(2i+1)(2i+3)(2i+5)}{(2i+2)(2i+4)(2i+6)} \mu^6 + \dots \right]; \\ \frac{1}{2} b^{(0)} = 1 + \frac{(\frac{1}{2})^2 \mu^2}{(1)^2} + \frac{(\frac{1}{2})^2 (\frac{1}{2} + 1)^2 \mu^4}{(1 \cdot 2)^2} + \dots + \frac{(\frac{1}{2})^2 (\frac{1}{2} + 1)^2 \dots (\frac{1}{2} + i-1)^2}{(1 \cdot 2 \cdot 3 \dots i)^2} \mu^{2i} \\ + \text{etc.}, \\ = 1 + (\frac{1}{2})^2 \mu^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \mu^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 \mu^6 + \dots \\ + \left(\frac{1 \cdot 3 \cdot 5 \dots 2i-1}{2 \cdot 4 \cdot 6 \dots 2i} \right)^2 \mu^{2i} + \text{etc.};$$

and $b^{(i)} = b^{(-i)}$.

[See for instance Tisserand 'Mec. Celeste.' 1. pp. 270-72].

$$\therefore V = -\frac{Ze^2}{r} + \frac{e^2}{r} \left[\left\{ (\frac{1}{2})^2 \mu^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \mu^4 + \dots \right\} \times n + \sum_{-\infty}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots 2i-1}{2 \cdot 4 \cdot 6 \dots 2i} \mu^i \cdot \right. \\ \left. \left\{ 1 + \frac{1}{2} \cdot \frac{2i+1}{2i+2} \mu^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{(2i+1)(2i+3)}{(2i+2)(2i+4)} \mu^4 + \dots \right\} \{ \cos i\theta + \cos i(\theta + \alpha) \right. \\ \left. + \dots + \cos i(\theta + n-1\alpha) \} \right]. \\ = -\frac{Ze^2}{r} + \frac{ne^2}{r} \left[1 + \left(\frac{1}{2} \right)^2 \mu^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \mu^4 + \dots \right] \\ + \sum_{-\infty}^{\infty} \frac{1 \cdot 3 \cdot 5 \dots 2i-1}{2 \cdot 4 \cdot 6 \dots 2i} \mu^i \left\{ 1 + \frac{1}{2} \cdot \frac{2i+1}{2i+2} \mu^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{(2i+1)(2i+3)}{(2i+2)(2i+4)} \mu^4 \right. \\ \left. + \dots \right\} \cos \left(i\theta + \frac{i}{n} (n-1)\pi \right) \cdot \frac{\sin i\pi}{\sin \frac{i\pi}{n}} \Bigg], \quad \text{putting in } \alpha = \frac{2\pi}{n}.$$

If $i \neq n$, or $2n$ or $3n$, etc.,

$$\sin i\pi / \sin \frac{i\pi}{n} = 0.$$

If $i = n$, $2n$, etc.,

$$\sin i\pi / \sin \frac{i\pi}{n} = \pm n,$$

according as n is odd or even. Hence

$$\cos \left\{ i\theta + \frac{i}{n} (n-1)\pi \right\} \frac{\sin i\pi}{\sin \frac{i\pi}{n}} = n \cos n\theta, n \cos 2n\theta, \text{ etc.},$$

whether n is even or odd.

We find then

$$\begin{aligned} V = & -\frac{Ze^2}{r} + \frac{ne^2}{r} \left[1 + \left(\frac{1}{2} \right)^2 \mu^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \mu^4 + \dots \right] \\ & + \frac{2e^2}{r} \cdot n \left[\frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} \mu^n \left(1 + \frac{1}{2} \cdot \frac{2n+1}{2n+2} \mu^2 \right. \right. \\ & + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{(2n+1)(2n+3)}{(2n+2)(2n+4)} \mu^4 + \dots \left. \right) \cos n\theta \\ & + \frac{1 \cdot 3 \cdot 5 \dots 4n-1}{2 \cdot 4 \cdot 6 \dots 4n} \mu^{2n} \left(1 + \frac{1}{2} \cdot \frac{4n+1}{4n+2} \mu^2 \right. \\ & + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{(4n+1)(4n+3)}{(4n+2)(4n+4)} \mu^4 + \dots \left. \right) \cos 2n\theta + \dots \left. \right] \\ = & -\frac{Ze^2}{r} + \frac{ne^2}{r} \left[1 + \left(\frac{1}{2} \right)^2 \mu^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \mu^4 + \dots \right] \\ & + \frac{2ne^2}{r} \left[f_n(r) \cos n\theta + f_{2n}(r) \cos 2n\theta + \dots \right] \end{aligned}$$

where

$$\begin{aligned} f_n(r) = & \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} \mu^n \left[1 + \frac{1}{2} \cdot \frac{2n+1}{2n+2} \mu^2 + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{(2n+1)(2n+3)}{(2n+2)(2n+4)} \mu^4 + \dots \right] \\ = & \mu^n \alpha_n + \mu^{n+2} \beta_n + \mu^{n+4} \gamma_n + \dots \end{aligned}$$

$$\text{if } \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n} \quad \beta_n = \frac{1}{2} \cdot \frac{2n+1}{2n+2}$$

$$\gamma_n = \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{(2n+1)(2n+3)}{(2n+2)(2n+4)} \alpha_n = \frac{3}{4} \cdot \frac{2n+3}{2n+4} \beta_n ; \text{ etc.}$$

as found out in (a). Similar values of $f_{2n}(r)$, $f_{3n}(r)$ etc.

The expression for this potential may be compared with that given by Sommerfeld.¹ This is

$$V = -\frac{Ze^2}{r} + \frac{ne^2}{r} \left[1 + \left(\frac{1}{2} \right)^2 \left(\frac{a}{r} \right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \left(\frac{a}{r} \right)^4 + \dots \right].$$

Thus Sommerfeld has got only the first term which is free from θ . The remaining terms involve both r and θ and cannot be neglected except in the exceptional circumstances, except when $\left(\frac{a}{r} \right)^n$ is very small, or r is large compared to a , i.e., for distant orbits like $2p$ or $3d^2$ or $2_s, 3_s$ orbits in Bohr's² newer notation. For the treatment of (*ms*) orbits, such approximations are not permissible. This is not allowable in atoms of smaller atomic weight like Lithium where n is small (2 in the case of Lithium), or where in the outer ring, there are more than one electron, e.g., in the case of the alkaline earths or elements of higher groups.

III

EQUATIONS OF MOTION OF THE VALENCY ELECTRON.

Suppose the n -electrons distributed at equal distances on the ring are all describing the same unperturbed circle of radius a with angular velocity ω (there being no mutual perturbations between them); and also that the valency electron describes a perturbed circle of mean radius b with normal angular velocity ω' . We define

$$\begin{aligned} \phi &= \omega t + \epsilon, \\ \chi &= \omega' t + \epsilon' + \sigma, \\ &= \theta + \phi, \\ OP &= r = b + \rho, \end{aligned}$$

¹ 'Atombau und Spektrallinien,' 2nd ed., *Zusätze und Ergänzungen* 10, p. 507.

² Ibid, Chap., VI.

³ Bohr—loc. cit., p. 20.

ϕ =angle which the radius vector to any one of the inner electrons makes with a line (OI₁) fixed in space,

θ =angle between the radiivectores to the outer electron at P and the inner electron above referred to.

χ =angle made by OP with the fixed line in space,

ρ is small in comparison to a , b .

σ is always a small angle.

Whence

$$\begin{aligned}\theta &= \chi - (\omega t + \epsilon), \\ &= (\omega' - \omega)t + (\epsilon' - \epsilon) + \sigma, \\ &= 1 + \sigma, \text{ say.}\end{aligned}$$

Equations of motion are :

$$\left. \begin{aligned}m \left[\frac{d^2 r}{dt^2} - r \left(\frac{d\chi}{dt} \right)^2 \right] &= - \frac{\partial V}{\partial r}, \\ m \cdot \left[\frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\chi}{dt} \right) \right] &= - \frac{1}{r} \frac{\partial V}{\partial \chi},\end{aligned} \right\}$$

or

$$\left. \begin{aligned}m \left[\frac{d^2 \rho}{dt^2} - 2b\omega \frac{d\sigma}{dt} - b\omega'^2 - \rho\omega'^2 \right] &= - \frac{\partial V}{\partial \rho}, \\ m \left[b^2 \frac{d^2 \sigma}{dt^2} + 2b\omega' \frac{d\rho}{dt} \right] &= - \frac{\partial V}{\partial \chi}.\end{aligned} \right\}$$

Remembering

$$\begin{aligned}V &= - \frac{Ze^2}{r} + \frac{ne^2}{r} \left[1 + \left(\frac{1}{2} \right)^2 \left(\frac{a}{r} \right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \left(\frac{a}{r} \right)^4 + \dots \right] \\ &\quad + \frac{2ne^2}{r} [f_1(r) \cos \theta + f_2(r) \cos 2\theta + \dots],\end{aligned}$$

we find for a electrically neutral atom, $Z=n+1$, and

$$\begin{aligned}- \frac{\partial V}{\partial r} &= - \frac{e^2}{r^2} + \frac{ne^2}{r^2} \left[\left(\frac{1}{2} \right)^2 a^2 \cdot \frac{3}{r^2} + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 a^4 \cdot \frac{5}{r^4} + \dots \right] \\ &\quad - 2ne^2 \left[\cos \theta \cdot \frac{d}{dr} \cdot r^{-2} f_1(r) + \cos 2\theta \cdot \frac{d}{dr} \cdot r^{-2} f_2(r) + \dots \right].\end{aligned}$$

Now

$$\frac{d}{dr} \cdot r^{-1} f_n(r) = -\frac{1}{r^2} \left\{ (n+1) a_n \left(\frac{a}{r} \right)^n + (n+3) \beta_n \left(\frac{a}{r} \right)^{n+2} + \dots \right\},$$

$$\frac{d}{dr} \cdot r^{-1} f_{2n}(r) = -\frac{1}{r^2} \left\{ (2n+1) a_{2n} \left(\frac{a}{r} \right)^{2n} + (2n+3) \beta_{2n} \left(\frac{a}{r} \right)^{2n+2} + \dots \right\};$$

etc.

Substituting we find

$$\begin{aligned} -\frac{\partial V}{\partial r} &= -\frac{e^2}{r^2} + \frac{ne^2}{r^2} \left[3 \cdot \left(\frac{1}{2} \right)^2 \cdot \left(\frac{a}{r} \right)^2 + 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \left(\frac{a}{r} \right)^4 + \dots \right] \\ &\quad + \frac{2ne^2}{r^2} \left[\left\{ (n+1) a_n \left(\frac{a}{r} \right)^n + (n+3) \beta_n \left(\frac{a}{r} \right)^{n+2} + \dots \right\} \cos n\theta \right. \\ &\quad \left. + \left\{ (2n+1) a_{2n} \left(\frac{a}{r} \right)^{2n} + (2n+3) \beta_{2n} \left(\frac{a}{r} \right)^{2n+2} + \dots \right\} \cos 2n\theta + \dots \right], \\ &= -\frac{e^2}{b^2} \left(1 + \frac{\rho}{b} \right)^{-2} + ne^2 \left[3 \cdot \left(\frac{1}{2} \right)^2 \frac{a^2}{b^4} \left(1 + \frac{\rho}{b} \right)^{-4} \right. \\ &\quad \left. + 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \frac{a^4}{b^6} \left(1 + \frac{\rho}{b} \right)^{-6} + \dots \right] \\ &\quad + 2ne^2 \left[\left\{ (n+1) a_n \frac{a^n}{b^{n+2}} \left(1 + \frac{\rho}{b} \right)^{-n-2} \right. \right. \\ &\quad \left. \left. + (n+3) \beta_n \frac{a^{n+2}}{b^{n+4}} \left(1 + \frac{\rho}{b} \right)^{-n-4} + \dots \right\} \cos n\theta \right. \\ &\quad \left. + \left\{ (2n+1) a_{2n} \frac{a^{2n}}{b^{2n+2}} \left(1 + \frac{\rho}{b} \right)^{-2n-2} \right. \right. \\ &\quad \left. \left. + (2n+3) \beta_{2n} \frac{a^{2n+2}}{b^{2n+4}} \left(1 + \frac{\rho}{b} \right)^{-2n-4} + \dots \right\} \cos 2n\theta + \text{etc.} \right]. \end{aligned}$$

Now $\theta = I + \sigma$; $\cos n\theta = \cos nI + n\sigma \sin nI$; etc. Substituting in the above the values of $\cos n\theta$, $\cos 2n\theta$, ..., we find, if $k = a/b$:—

$$\begin{aligned}
 -\frac{\partial V}{\partial r} = & -\frac{e^2}{b^3} \left[1 - n \left\{ 3 \left(\frac{1}{2} \right)^n k^2 + 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^n k^4 + \dots \right\} \right. \\
 & + \frac{2ne^2}{b^3} \left[\{ (n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots \} \cos nI \right. \\
 & + \{ (2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots \} \cos 2nI + \dots] \\
 & + \frac{e^2 p}{b^3} \left[2 - n \left\{ 3 \cdot 4 \left(\frac{1}{2} \right)^n k^2 + 6 \cdot 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^n k^4 + \dots \right\} \right. \\
 & - 2n \left\{ (n+2)(n+1)a_n k^n + (n+4)(n+3)\beta_n k^{n+2} + \dots \right\} \cos nI \\
 & + \left\{ (2n+2)(2n+1)a_{2n} k^{2n} + (2n+4)(2n+3)\beta_{2n} k^{2n+2} + \dots \right\} \cos 2nI \\
 & + \dots \left. \right] + \frac{2n^2 e^2}{b^3} \sigma \left[\{ (n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots \} \sin nI \right. \\
 & + 2 \{ (2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots \} \sin 2nI + \dots \left. \right]. \\
 -\frac{\partial V}{\partial \chi} = & -\frac{2nn^2}{r} \left[f_n(r) \cdot \frac{\partial}{\partial \chi} \cos n(\chi - \phi) + f_{2n}(r) \cdot \frac{\partial}{\partial \chi} \cos 2n(\chi - \phi) + \dots \right] \\
 = & \frac{2n^3 e^2}{r} \left[f_n(r) \sin n(\chi - \phi) + 2f_{2n}(r) \sin 2n(\chi - \phi) + \dots \right] \\
 = & \frac{2n^3 e^2}{r} \left[f_n(r) \sin n(1 + \sigma) + 2f_{2n}(r) \sin 2n(1 + \sigma) + \dots \right].
 \end{aligned}$$

Now

$$\sin n(1 + \sigma) = \sin nI + n\sigma \cos nI, \text{ etc.}$$

Substituting we find as before

$$\begin{aligned}
 -\frac{\partial V}{\partial \chi} = & \frac{2n^3 e^2}{b} \left[(a_n k^n + \beta_n k^{n+2} + \dots) \sin nI \right. \\
 & + 2(a_{2n} k^{2n} + \beta_{2n} k^{2n+2} + \dots) \sin 2nI + \dots \left. \right]
 \end{aligned}$$

$$\begin{aligned}
& + \frac{2n^2 e^2}{b} \sigma [(a_n k^n + \beta_n k^{n+2} + \dots) \cos nI \\
& + 2^2 \cdot (a_{2n} k^{2n} + \beta_{2n} k^{2n+2} + \dots) \cos 2nI + \dots] \\
& - \frac{2n^2 e^2}{b^2} \rho [\{(n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots\} \sin nI \\
& + 2\{(2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots\} \sin 2nI + \dots].
\end{aligned}$$

The equations of motion can be written thus

$$\begin{aligned}
\frac{d^2 \rho}{dt^2} - 2b\omega' \frac{d\sigma}{dt} - b\omega'^2 - \rho\omega'^2 \\
= - \frac{e^2}{mb^2} [1 - n \{3 \cdot \left(\frac{1}{2}\right)^2 k^2 + 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \dots\}] \\
+ \frac{2n^2 e^2}{b^2 m} [\{(n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots\} \cos nI + \dots] \\
+ \frac{e^2 \rho}{b^2 m} [2 - n \{3 \cdot 4 \left(\frac{1}{2}\right)^2 k^2 - 6 \cdot 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \dots\} \\
- 2n \{(n+2)(n+1)a_n k^n + (n+4)(n+3)\beta_n k^{n+2} + \dots\} \cos nI + \dots] \\
+ \frac{2n^2 e^2}{b^2 m} \sigma [\{(n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots\} \sin nI \\
+ 2\{(2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots\} \sin 2nI + \dots] \quad \dots \quad 1.
\end{aligned}$$

$$\begin{aligned}
b \frac{d^2 \sigma}{dt^2} + 2\omega' \frac{d\rho}{dt} = \frac{2n^2 e^2}{b^2 m} [(a_n k^n + \beta_n k^{n+2} + \dots) \sin nI \\
+ 2(a_{2n} k^{2n} + \beta_{2n} k^{2n+2} + \dots) \sin 2nI + \dots] + \frac{2n^2 e^2}{b^2 m} \sigma [(a_n k^n \\
+ \beta_n k^{n+2} + \dots) \cos nI + 2^2 \cdot (a_{2n} k^{2n} + \beta_{2n} k^{2n+2} + \dots) \cos 2nI + \dots, \\
\cdot \frac{2n^2 e^2}{b^2 m} \rho [\{(n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots\} \sin nI \\
+ 2\{(2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots\} \sin 2nI + \dots] \quad \dots \quad 11.
\end{aligned}$$

$$\text{Now} \quad I = (\omega' - \omega)t + (\epsilon' - \epsilon)$$

$$\therefore \frac{d}{dt} = (\omega' - \omega) \frac{d}{dI}; \quad \frac{d^2}{dt^2} = (\omega' - \omega)^2 \frac{d^2}{dI^2}.$$

• Equations I, II can then be written in the form

$$\frac{d^2 \rho}{dI^2} - 2b \frac{\omega'}{\omega' - \omega} \frac{d\sigma}{dI} - b \frac{\omega'^2}{(\omega' - \omega)^2}$$

$$- \rho \frac{\omega'^2}{(\omega' - \omega)^2} = \frac{1}{(\omega' - \omega)^2} \quad [\text{idem of Equation I}] \quad \dots \quad \text{I'}$$

$$\frac{d^2 \sigma}{dI^2} + 2 \frac{\omega'}{\omega' - \omega} \frac{d\rho}{dI} = \frac{1}{b(\omega' - \omega)^2} \quad [\text{idem of Equation II}] \quad \dots \quad \text{II'}$$

For the sake of homogeneity write $b\rho$ for ρ , so that

$$r = b(1 + \rho).$$

Hence, we have

$$\frac{d^2 \rho}{dI^2} - 2 \frac{\omega'}{\omega' - \omega} \frac{d\sigma}{dI} - \frac{\omega'^2}{(\omega' - \omega)^2} - \rho \frac{\omega'^2}{(\omega' - \omega)^2} = \frac{1}{b(\omega' - \omega)^2} \quad [\text{idem}] \quad \dots \quad \text{I''}$$

$$\frac{d^2 \sigma}{dI^2} + 2 \frac{\omega'}{\omega' - \omega} \frac{d\rho}{dI} = \frac{1}{b(\omega' - \omega)^2} \quad [\text{idem}] \quad \dots \quad \text{II''}$$

Now put $-\omega'/\omega - \omega' = \nu$, so that

$$\frac{1}{b^2(\omega - \omega')^2} = \frac{\nu^2}{b^2\omega'^2} = \frac{\nu^2}{e^2/mc^2}.$$

Because

$$mb\omega'^2 = \frac{\partial V}{\partial r} = \frac{e^2}{b^2}$$

approximately, as can be seen from the value of $-\frac{\partial V}{\partial r}$ found above.

If $(\rho'', \sigma'', \rho', \sigma')$ stand for

$$\left(\frac{d^2 \rho}{dI^2}, \frac{d^2 \sigma}{dI^2}, \frac{d\rho}{dI}, \frac{d\sigma}{dI} \right)$$

respectively, the equations I', II' can be re-written in the form thus

$$\begin{aligned} \rho'' - 2\nu\sigma' - \nu^2 - \rho\nu^2 = & -\nu^2 \left[1 - n \left\{ 3 \cdot \left(\frac{1}{2} \right)^2 k^2 + 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \dots \right\} \right] \\ & + 2n\nu^2 \left[\{ (n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots \} \cos nI \right. \\ & \left. + \{ (2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots \} \cos 2nI + \dots \right] \\ & + \nu^2 \rho \left[2 - n \left\{ 3 \cdot 4 \cdot \left(\frac{1}{2} \right)^2 k^2 + 6 \cdot 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \dots \right\} \right. \\ & \left. - 2n \left\{ (n+2)(n+1)a_n k^n + (n+4)(n+3)\beta_n k^{n+2} + \dots \right\} \cos nI \right. \\ & \left. + \left\{ (2n+2)(2n+1)a_{2n} k^{2n} + (2n+4)(2n+3)\beta_{2n} k^{2n+2} + \dots \right\} \cos 2nI \right. \\ & \left. + \dots \right] + 2n^2 \nu^2 \sigma \left[\{ (n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots \} \sin nI \right. \\ & \left. + 2 \{ (2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots \} \sin 2nI + \dots \right] \dots \quad (3) \end{aligned}$$

$$\begin{aligned} \sigma'' + 2\nu\rho' = & 2n^2 \nu^2 \left[(a_n k^n + \beta_n k^{n+2} + \dots) \sin nI + 2(a_{2n} k^{2n} + \beta_{2n} k^{2n+2} \right. \\ & \left. + \dots) \sin 2nI + \dots \right] + 2n^3 \nu^3 \sigma \left[(a_n k^n + \beta_n k^{n+2} + \dots) \cos nI \right. \\ & \left. + 2(a_{2n} k^{2n} + \beta_{2n} k^{2n+2} + \dots) \cos 2nI + \dots \right] \\ & - 2n^2 \nu^2 \rho \left[\{ (n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots \} \sin nI \right. \\ & \left. + 2 \{ (2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots \} \sin 2nI + \dots \right] \dots \quad (4) \end{aligned}$$

Equations (3) and (4) can be briefly expressed thus :

$$\begin{aligned} \rho'' - 2\nu\sigma' + (\odot_{1,0} + \odot_{1,1} \cos nI + \odot_{1,2} \cos 2nI + \dots) \rho \\ + (\odot_{2,1} \sin nI + \odot_{2,2} \sin 2nI + \dots) \sigma \\ = \odot_{3,0} + \odot_{3,1} \cos nI + \odot_{3,2} \cos 2nI + \dots \quad \dots \quad (5) \end{aligned}$$

$$\begin{aligned} \sigma'' + 2\nu\rho' + (\odot_{4,1} \sin nI + \odot_{4,2} \sin 2nI + \dots) \rho \\ + (\odot_{5,1} \cos nI + \odot_{5,2} \cos 2nI + \dots) \sigma \\ = \odot_{6,1} \sin nI + \odot_{6,2} \sin 2nI + \dots \quad \dots \quad (6) \end{aligned}$$

The values of \odot 's are given below :

$$\odot_{1,0} = v^2 \left[-3 + n \left\{ 3 \cdot 4 \cdot \left(\frac{1}{2} \right)^2 k^2 + 6 \cdot 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \dots \right\} \right];$$

$$\odot_{1,1} = 2nv^2 \left[(n+2)(n+1)a_n k^n + (n+4)(n+3)\beta_n k^{n+2} + \dots \right];$$

$$\odot_{1,2} = 2nv^2 \left[(2n+2)(2n+1)a_{2n} k^{2n} + (2n+4)(2n+3)\beta_{2n} k^{2n+2} + \dots \right];$$

$$\odot_{2,1} = -2nv^2 \left[(n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots \right];$$

$$\odot_{2,2} = -4nv^2 \left[(2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots \right];$$

$$\odot_{3,0} = nv^2 \left[3 \cdot \left(\frac{1}{2} \right)^2 k^2 + 5 \cdot \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \dots \right];$$

$$\odot_{3,1} = 2nv^2 \left[(n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots \right];$$

$$\odot_{3,2} = 2nv^2 \left[(2n+2)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots \right];$$

$$\odot_{4,1} = 2n^2 v^2 \left[(n+1)a_n k^n + (n+3)\beta_n k^{n+2} + \dots \right];$$

$$\odot_{4,2} = 4n^2 v^2 \left[(2n+1)a_{2n} k^{2n} + (2n+3)\beta_{2n} k^{2n+2} + \dots \right];$$

$$\odot_{5,1} = -2n^3 v^2 \left[a_n k^n + \beta_n k^{n+2} + \dots \right];$$

$$\odot_{5,2} = -8n^3 v^2 \left[a_{2n} k^{2n} + \beta_{2n} k^{2n+2} + \dots \right];$$

$$\odot_{6,1} = 2n^2 v^2 \left[a_n k^n + \beta_n k^{n+2} + \dots \right];$$

$$\odot_{6,2} = 4n^2 v^2 \left[a_{2n} k^{2n} + \beta_{2n} k^{2n+2} + \dots \right];$$

and so on. Since v, k are fractional we see that among the \odot 's, only $\odot_{1,0}, \odot_{3,0}$ are larger than others (provided $n > 2$) by an appreciable amount.

IV

SOLUTION OF THE EQUATIONS.

(a) The Complementary Function

The equations

$$\rho'' - 2v\rho' + \rho \sum_{r=0} \odot_{1,r} \cos rnI + \sigma \sum_{r=1} \odot_{2,r} \sin rnI = 0$$

... [A]

$$\sigma'' + 2v\rho' + \rho \sum_{r=1} \odot_{4,r} \sin rnI + \sigma \sum_{r=1} \odot_{5,r} \cos rnI = 0$$

are homogeneous linear differential equations with periodic coefficients, and the solutions, as will be seen, are quasi-periodic solutions of the type $e^{c1} \phi(1)$, where $\phi(1)$ is a periodic function having the same period as the coefficients in the above equations and the parameter c is the factor of quasi-periodicity. The crux of the problem centres round a complete determination of this important parameter. Equations of this nature with *one dependent variable* have been discussed by Hill,¹ Young,² Ince,³ Baker⁴ and Whittaker;⁵ and were first used by Hill in his classical studies on the perturbations of the moon. The present equation is, however, of a more general type as it involves *two dependent variables* instead of *one* as in Hill's equation: a modification of their methods first introduced by Goldsbrough⁶ will be employed here for an integral of these equations, as Hill's general analysis involves an evaluation of infinite determinant in c , which is unmanageable.

Suppose

$$\rho = e^{c1} \cdot A,$$

$$\sigma = e^{-1} \cdot X,$$

where A and X are purely periodic functions of period 2π . On substituting in equations (A) we find, since

$$\frac{d^2}{d1^2} (e^{c1} A) = e^{c1} A'' + 2ce^{c1} A' + c^2 e^{c1} A,$$

$$\frac{d^2}{d1^2} (e^{c1} X) = e^{c1} X'' + 2ce^{c1} X' + c^2 e^{c1} X,$$

$$\frac{d}{d1} (e^{c1} A) = e^{c1} A' + ce^{c1} A,$$

$$\frac{d}{d1} (e^{c1} X) = e^{c1} X' + ce^{c1} X.$$

¹ 'Acta Mathematica,' Vol. VIII; Whittaker's 'Modern Analysis.'

² 'Proc. Edin. Math. Soc.,' XXXII, p. 81.

³ 'M. N. R. A. S.,' LXXV, 5, p. 436.

⁴ 'Phil. Trans. A.,' Vol. 216, p. 129.

⁵ 'Proc. Inter. Congress. Math.,' Vol. I, 1912; 'Proc. Edin. Math. Soc.,' XXXII, p. 76; 'Modern Analysis.'

⁶ 'Phil. Trans.' Vol. 222, 1922.

$$\left. \begin{aligned}
 c^2 A + 2cA' + A'' - 2\nu(cX + X') + A \sum_{r=0}^{\infty} \odot_{1,r} \cos rnI \\
 + X \sum_{r=1}^{\infty} \odot_{2,r} \sin rnI = 0 \\
 c^2 X + 2cX' + X'' + 2\nu(cA + A') + A \sum_{r=1}^{\infty} \odot_{3,r} \sin rnI \\
 + X \sum_{r=1}^{\infty} \odot_{4,r} \cos rnI = 0
 \end{aligned} \right\} \dots [B]$$

As a general case of Whittaker's solution (Proc. Edin. Math. Soc., loc. cit. p. 77) of Mathew's differential equation in periodic functions let us assume series in \odot 's and multiples of \odot 's having coefficients with period 2π , i.e.

$$\begin{aligned}
 A = A_0 \sin(\lambda nI - \tau) + \sum \sum A_{r,s} \odot_{r,s} \\
 + \sum \sum \sum \sum B_{r,s,p,q} \odot_{r,s} \odot_{p,q} + \dots \dots \dots [E-1]
 \end{aligned}$$

$$\begin{aligned}
 X = X_0 \cos(\lambda nI - \tau) + \sum \sum X_{r,s} \odot_{r,s} \\
 + \sum \sum \sum \sum Y_{r,s,p,q} \odot_{r,s} \odot_{p,q} + \dots \dots \dots [E-2]
 \end{aligned}$$

Here A_0 , X_0 are arbitrary constants, λ is an arbitrary integer and τ a parameter which will be defined presently. As usual let

$$c = \sum \sum c_{r,s} \odot_{r,s} + \sum \sum \sum \sum d_{r,s,p,q} \odot_{r,s} \odot_{p,q} + \dots$$

where the coefficients of \odot 's and multiples of \odot 's are functions of n , λ and τ .

Substitute these values of A , X and c in equations [B]. Thus

$$\begin{aligned}
 & \{ \sum \sum c_{r,s} \odot_{r,s} + \dots \} \{ A_0 \sin(\lambda nI - \tau) + \sum \sum A_{r,s} \odot_{r,s} + \dots \} \\
 & + 2 \{ \sum \sum c_{r,s} \odot_{r,s} + \dots \} \{ A_0 n \lambda \cos(\lambda nI - \tau) + \sum \sum A'_{r,s} \odot_{r,s} + \dots \} \\
 & + \{ -A_0 \lambda^2 n^2 \sin(\lambda nI - \tau) + \sum \sum A''_{r,s} \odot_{r,s} + \dots \} \\
 & - 2\nu \{ \sum \sum c_{r,s} \odot_{r,s} + \dots \} \{ X_0 \cos(\lambda nI - \tau) + \sum \sum X_{r,s} \odot_{r,s} + \dots \} \\
 & + \{ -X_0 \lambda n \sin(\lambda nI - \tau) + \sum \sum X'_{r,s} \odot_{r,s} + \dots \} \\
 & + \{ A_0 \sin(\lambda nI - \tau) + \sum \sum A_{r,s} \odot_{r,s} + \dots \} \sum_{r=0}^{\infty} \odot_{1,r} \cos rnI \\
 & + \{ X_0 \cos(\lambda nI - \tau) + \sum \sum X_{r,s} \odot_{r,s} + \dots \} \sum_{r=1}^{\infty} \odot_{2,r} \sin rnI = 0 \dots [C]
 \end{aligned}$$

and

$$\begin{aligned}
 & \{ \sum \sum c_{r,s} \odot_{r,s} + \dots \}^2 \{ X_0 \cos(\lambda n I - \tau) + \sum \sum X_{r,s} \odot_{r,s} + \dots \} \\
 & + 2 \{ \sum \sum c_{r,s} \odot_{r,s} + \dots \} \{ -X_0 \lambda n \sin(\lambda n I - \tau) + \sum \sum X'_{r,s} \odot_{r,s} + \dots \} \\
 & + \{ -\lambda^2 n^2 X_0 \cos(\lambda n I - \tau) + \sum \sum X''_{r,s} \odot_{r,s} + \dots \} \\
 & + 2v \{ \sum \sum c_{r,s} \odot_{r,s} + \dots \} \{ A_0 \sin(\lambda n I - \tau) + \sum \sum A_{r,s} \odot_{r,s} + \dots \} \\
 & + \{ A_0 \lambda n \cos(\lambda n I - \tau) + \sum \sum A'_{r,s} \odot_{r,s} + \dots \} \\
 & + \{ A_0 \sin(\lambda n I - \tau) + \sum \sum A_{r,s} \odot_{r,s} + \dots \} \sum_{r=1}^{\infty} \odot_{1,r} \sin r n I \\
 & + \{ X_0 \cos(\lambda n I - \tau) + \sum \sum X_{r,s} \odot_{r,s} + \dots \} \sum_{r=1}^{\infty} \odot_{3,r} \cos r n I = 0 \quad [D]
 \end{aligned}$$

First equate to zero those terms not involving any \odot . Since in the above series we have not included $\odot_{1,0}$ which is large compared with the others, we shall get $\odot_{1,0}$ on equating, in the \odot -independent terms.

Thus

$$\{ (\odot_{1,0} - \lambda^2 n^2) A_0 + 2v \lambda n X_0 \} \sin(\lambda n I - \tau) = 0$$

$$\{ 2v \lambda n A_0 + (0 - \lambda^2 n^2) X_0 \} \cos(\lambda n I - \tau) = 0.$$

Now A_0, X_0 being assumed not equal to zero, on eliminating A_0, X_0 we get $\odot_{1,0} = \lambda^2 n^2 - 4v^2$. In general the given value of $\odot_{1,0}$ will not satisfy this equation: for any integral value of λ . Suppose $a_{1,0}$ is a quantity which satisfies $a_{1,0} = \lambda^2 n^2 - 4v^2$, for some integral value of λ , where $a_{1,0}$, of course, slightly differs from $\odot_{1,0}$.

Now assume

$$\odot_{1,0} = a_{1,0} + \sum \sum a_{r,s} \odot_{r,s} + \sum \sum \sum b_{r,s,p,q} \odot_{r,s} \odot_{p,q} + \dots \dots [E]$$

To determine all the unknown coefficients we should impose two conditions.

(i) The series for A does not contain the term $\cos(\lambda n I - \tau)$: in fact in this really constitutes the definition of the parameter τ and the possibility of obtaining series which remains convergent for all real values of τ depends upon our choosing τ in this way.

(ii) The solutions for A and X must be purely periodic with period 2π (i.e., no part of the exponent shall appear in the periodic series).

Further, these conditions will determine uniquely the undetermined coefficients in the series for c and $\odot_{1,0}$.

On substituting the assumed series for A , X , c and $\odot_{1,0}$ in equation [B] and equating to zero the terms involving $\odot_{1,r}$, as from equations [C] and [D] we find

$$2c_{1,r}\lambda n A_0 \cos(\lambda n I - \tau) + A''_{1,r} - 2\nu c_{1,r} X_0 \cos(\lambda n I - \tau) - 2\nu X'_{1,r} + a_{1,0} A_{1,r} + a_{1,r} A_0 \sin(\lambda n I - \tau) + A_0 \cos r n I \sin(\lambda n I - \tau) = 0 \dots \quad [C \cdot 1]$$

$$-2c_{1,r}\lambda n X_0 \sin(\lambda n I - \tau) + X''_{1,r} + 2\nu c_{1,r} A_0 \sin(\lambda n I - \tau) + 2\nu A'_{1,r} = 0 \dots \quad [D \cdot 1]$$

Now

$$\cos r n I \sin(\lambda n I - \tau) = \frac{1}{2} [\sin\{n(r+\lambda)I - \tau\} + \sin\{n(\lambda-r)I - \tau\}]$$

When $r \neq \lambda$ or 2λ , it is clear that

$$c_{1,r} = 0, \quad a_{1,r} = 0.$$

Equations [C·1], [D·1] reduce to

$$A''_{1,r} - 2\nu X'_{1,r} + a_{1,0} A_{1,r} + \frac{1}{2} A_0 [\sin\{n(\lambda+r)I - \tau\} + \sin\{n(\lambda-r)I - \tau\}] = 0,$$

$$X''_{1,r} + 2\nu A'_{1,r} = 0,$$

Putting

$$A_{1,r} = p \sin\{(\lambda+r)nI - \tau\} + q \sin\{(\lambda-r)nI - \tau\},$$

$$X_{1,r} = p' \cos\{(\lambda+r)nI - \tau\} + q' \cos\{(\lambda-r)nI - \tau\},$$

where (p, q, p', q') are constants and solving as usual we find

$$A_{1,r} = \frac{A_0 \sin\{(\lambda+r)nI - \tau\}}{2rn(2\lambda n + rn)} - \frac{A_0 \sin\{(\lambda-r)nI - \tau\}}{2rn(2\lambda n - rn)},$$

$$X_{1,r} = \frac{A_0 r \cos\{(\lambda+r)nI - \tau\}}{rn(\lambda n + rn)(2\lambda n + rn)} - \frac{A_0 r \cos\{(\lambda-r)nI - \tau\}}{rn(\lambda n - rn)(2\lambda n - rn)}.$$

In the special case, when $r = \lambda$, we have

$$\left. \begin{aligned} A''_{1,\lambda} - 2\nu X'_{1,\lambda} + a_{1,0} A_{1,\lambda} + \frac{1}{2} A_0 [\sin(2\lambda n I - \tau) - \sin \tau] &= 0 \\ X''_{1,\lambda} + 2\nu A'_{1,\lambda} &= 0 \end{aligned} \right\} [F \cdot 1]$$

These give

$$c_{1,\lambda}=0, \quad u_{1,\lambda}=0$$

$$A_{1,\lambda} = \frac{A_0 \sin(2\lambda l - \tau)}{6\lambda^2 n^2} + \frac{A_0 \sin \tau}{a_{1,0}};$$

$$X_{1,\lambda} = \frac{A_0 \nu \cos(2\lambda l - \tau)}{6\lambda^2 n^2}.$$

When $r=2\lambda$, equations [C·1], [D·1] can be written in the form :

$$\begin{aligned} 2c_{1,2\lambda} \lambda n A_0 \cos(\lambda n l - \tau) + A''_{1,2\lambda} - 2\nu c_{1,2\lambda} X_0 \cos(\lambda n l - \tau) \\ - 2\nu X'_{1,2\lambda} + a_{1,0} A_{1,2\lambda} + a_{1,2\lambda} A_0 \sin(\lambda n l - \tau) + \frac{1}{2} A_0 [\sin(3\lambda n l - \tau) \\ - \sin(\lambda n l - \tau) \cos 2\tau - \cos(\lambda n l - \tau) \sin 2\tau] = 0 \end{aligned} \quad \dots \quad [C'2]$$

$$\begin{aligned} -2c_{1,2\lambda} \lambda n X_0 \sin(\lambda n l - \tau) + X''_{1,2\lambda} \\ + 2\nu c_{1,2\lambda} A_0 \sin(\lambda n l - \tau) + 2\nu A'_{1,2\lambda} = 0 \end{aligned} \quad \dots \quad [D'2]$$

To obtain $a_{1,2\lambda}$ we collect the $\sin(\lambda n l - \tau)$ terms, thus

$$\left. \begin{aligned} A''_{1,2\lambda} - 2\nu X'_{1,2\lambda} + a_{1,0} A_{1,2\lambda} \\ + A_0 \sin(\lambda n l - \tau) [a_{1,2\lambda} - \frac{1}{2} \cos 2\tau] = 0 \\ X''_{1,2\lambda} + 2\nu A'_{1,2\lambda} = 0 \end{aligned} \right\} \quad [E'2]$$

whence we get

$$a_{1,2\lambda} = \frac{1}{2} \cos 2\tau,$$

taking solutions for

$$A_{1,2\lambda}, X_{1,2\lambda}$$

of the form

$$p \sin(\lambda n l - \tau), q \cos(\lambda n l - \tau)$$

respectively and remembering

$$a_{1,0} = \lambda^2 n^2 - 4\nu^2.$$

To obtain $c_{1, 2\lambda}$, since A must not contain $\cos(\lambda n I - \tau)$ term (X may contain), we get from the equations

$$\left. \begin{aligned} (2c_{1, 2\lambda} \lambda n A_0 - 2\nu c_{1, 2\lambda} X_0 - \frac{1}{2} A_0 \sin 2\tau) \cos(\lambda n I - \tau) - 2\nu X'_{1, 2\lambda} &= 0 \\ (-2c_{1, 2\lambda} \lambda n X_0 + 2\nu c_{1, 2\lambda} A_0) \sin(\lambda n I - \tau) + X''_{1, 2\lambda} &= 0 \end{aligned} \right\} [F.3]$$

$$c_{1, 2\lambda} = \frac{1}{4} \frac{\sin 2\tau}{\lambda n^3}$$

remembering

$$\lambda n X_0 = 2\nu A_0 \text{ and } a_{1,0} = \lambda^2 n^2 - 4\nu^2.$$

Substituting this value of $c_{1, 2\lambda}$ in either of the equations [F.3]

we find

$$X_{1, 2\lambda} = -\frac{\nu A_0 \sin 2\tau \sin(\lambda n I - \tau)}{2\lambda^3 n^3}.$$

To obtain solutions of $A_{1, 2\lambda}$, $X_{1, 2\lambda}$ in $\sin(3\lambda n I - \tau)$, $\cos(3\lambda n I - \tau)$ terms write

$$\left. \begin{aligned} A''_{1, 2\lambda} - 2\nu X'_{1, 2\lambda} + a_{1,0} A_{1, 2\lambda} + \frac{1}{2} A_0 \sin(3\lambda n I - \tau) &= 0 \\ X''_{1, 2\lambda} + 2\nu A'_{1, 2\lambda} &= 0 \end{aligned} \right\} \dots [F.4]$$

Assume

$$A_{1, 2\lambda} = p \sin(3\lambda n I - \tau),$$

$$X_{1, 2\lambda} = q \cos(3\lambda n I - \tau).$$

Substituting in [F.4] and solving in p and q we find

$$p = \frac{1}{16} \frac{A_0}{\lambda^3 n^3}; \quad q = \frac{1}{24} \frac{\nu A_0}{\lambda^3 n^3}$$

i.e., the particular solutions arising from the term $\frac{1}{2} A_0 \sin(3\lambda n I - \tau)$ are

$$A_{1, 2\lambda} = \frac{1}{16} \frac{A_0}{\lambda^3 n^3} \sin(3\lambda n I - \tau).$$

$$X_{1, 2\lambda} = \frac{1}{24} \frac{\nu A_0}{\lambda^3 n^3} \cos(3\lambda n I - \tau).$$

or, the complete solution for $X_{1, 2\lambda}$ is

$$X_{1, 2\lambda} = -\frac{1}{2} \frac{\nu A_0}{\lambda^3 n^3} \sin 2\tau \sin(\lambda n I - \tau) + \frac{1}{24} \frac{\nu A_0}{\lambda^3 n^3} \cos(3\lambda n I - \tau).$$

Proceeding in the same mechanical process we can find out the other coefficients of \odot 's in the series for Λ , X , and the undetermined constants in the series for c and $\odot_{1,0}$. We write down the complete results thus.

Terms involving argument $\odot_{1,r}$:

$$c_{1,r} = 0, \quad a_{1,r} = 0,$$

$$A_{1,r} = \frac{A_0 \sin\{n(\lambda+r)I - \tau\}}{2rn(2\lambda n + rn)} - \frac{A_0 \sin\{n(\lambda-r)I - \tau\}}{2rn(2\lambda n - rn)};$$

$$X_{1,r} = \frac{A_0 \nu \cos\{n(\lambda+r)I - \tau\}}{rn(\lambda n + rn)(2\lambda n + rn)} - \frac{A_0 \nu \cos\{n(\lambda-r)I - \tau\}}{rn(\lambda n - rn)(2\lambda n - rn)}.$$

Terms involving argument $\odot_{1,\lambda}$:

$$c_{1,\lambda} = 0, \quad a_{1,\lambda} = 0.$$

$$A_{1,\lambda} = \frac{A_0 \sin(2\lambda n I - \tau)}{6\lambda^3 n^3} + \frac{A_0 \sin \tau}{2a_{1,\nu}};$$

$$X_{1,\lambda} = \frac{A_0 \nu \cos(2\lambda n I - \tau)}{6\lambda^3 n^3}.$$

Terms involving argument $\odot_{1, 2\lambda}$:

$$c_{1, 2\lambda} = \frac{1}{4} \frac{\sin 2\tau}{\lambda n}, \quad a_{1, 2\lambda} = \frac{1}{2} \cos 2\tau.$$

$$A_{1, 2\lambda} = \frac{1}{16} \frac{A_0}{\lambda^3 n^3} \sin(3\lambda n I - \tau);$$

$$X_{1, 2\lambda} = -\frac{1}{2} \frac{\nu A_0}{\lambda^3 n^3} \sin 2\tau \sin(\lambda n I - \tau) + \frac{1}{24} \frac{\nu A_0}{\lambda^3 n^3} \cos(3\lambda n I - \tau).$$

Terms involving argument $\odot_{2,r}$:

$$c_{2,r}=0, \quad a_{2,r}=0.$$

$$A_{2,r} = \frac{X_0 \sin\{n(\lambda+r)I-\tau\}}{2rn(2\lambda n+rn)} + \frac{X_0 \sin\{n(\lambda-r)I-\tau\}}{2rn(2\lambda n-rn)} ;$$

$$X_{2,r} = \frac{X_0 v \cos\{n(\lambda+r)I-\tau\}}{rn(\lambda n+rn)(2\lambda n+rn)} + \frac{X_0 v \cos\{n(\lambda-r)I-\tau\}}{rn(\lambda n-rn)(2\lambda n-rn)} .$$

Terms involving argument $\odot_{2,\lambda}$:

$$c_{2,\lambda}=0, \quad a_{2,\lambda}=0.$$

$$A_{2,\lambda} = \frac{X_0 \sin(2\lambda n I - \tau)}{6\lambda^3 n^3} - \frac{X_0 \sin \tau}{2a_{1,0}} ;$$

$$X_{2,\lambda} = \frac{X_0 v \cos(2\lambda n I - \tau)}{6\lambda^3 n^3} .$$

Terms involving argument $\odot_{2,2\lambda}$:

$$c_{2,2\lambda} = -\frac{v \sin 2\tau}{2\lambda^3 n^3}, \quad a_{2,2\lambda} = \frac{-v \cos 2\tau}{\lambda n} ,$$

$$A_{2,2\lambda} = \frac{1}{16} \frac{X_0 \sin(3\lambda n I - \tau)}{\lambda^3 n^3} ;$$

$$X_{2,2\lambda} = \frac{X_0 v \sin 2\tau \sin(\lambda n I - \tau)}{2\lambda^3 n^3} + \frac{1}{24} \frac{v X_0 \cos(3\lambda n I - \tau)}{\lambda^3 n^3} .$$

Terms involving argument $\odot_{4,r}$:

$$c_{4,r}=0, \quad a_{4,r}=0.$$

$$A_{4,r} = \frac{-v A_0 \sin\{n(\lambda+r)I-\tau\}}{rn(\lambda n+rn)(2\lambda n+rn)} - \frac{v A_0 \sin\{n(\lambda-r)I-\tau\}}{rn(\lambda n-rn)(2\lambda n-rn)} ;$$

$$X_{4,r} = \frac{A_0 \{a_{1,0} - (\lambda n + rn)^2\} \cos\{n(\lambda+r)I-\tau\}}{2rn(\lambda n+rn)^2(2\lambda n+rn)} \\ + \frac{A_0 \{a_{1,0} - (\lambda n - rn)^2\} \cos\{n(\lambda-r)I-\tau\}}{2rn(\lambda n-rn)^2(2\lambda n-rn)} ;$$

Terms involving argument $\odot_{4,\lambda}$:

$$c_{4,\lambda}=0, \quad a_{4,\lambda}=0.$$

$$A_{4,\lambda} = \frac{-\nu A_0 \sin(2\lambda n I - \tau)}{6\lambda^3 n^3};$$

$$X_{4,r} = \frac{A_0(a_{1,0} - 4\lambda^2 n^2) \cos(2\lambda n I - \tau)}{24\lambda^4 n^4}.$$

[here in addition $\cos \tau = 0$, in order to avoid the existence of I occurring explicitly in $A_{4,\lambda}$.]

Terms involving argument $\odot_{4,2\lambda}$:

$$c_{4,2\lambda} = \frac{-\nu \sin 2\tau}{2\lambda^2 n^2}, \quad a_{4,2\lambda} = -\frac{\nu \cos 2\tau}{\lambda n}$$

$$A_{4,2\lambda} = -\frac{1}{24} \frac{A_0 \nu \sin(3\lambda n I - \tau)}{\lambda^3 n^3};$$

$$X_{4,2\lambda} = \frac{1}{2} \frac{A_0 \cos 2\tau \cos(\lambda n I - \tau)}{\lambda^2 n^2} - \frac{A_0 \sin 2\tau (a_{1,0} + 2\nu^2)}{2\lambda^3 n^3} \\ + \frac{1}{16} \frac{A_0 (\nu^2 + 2\lambda^2 n^2)}{\lambda^4 n^4} \cos(3\lambda n I - \tau).$$

Terms involving argument $\odot_{5,r}$:

$$c_{5,r}=0, \quad a_{5,r}=0.$$

$$A_{5,r} = \frac{X_0 \nu \sin\{n(\lambda+r)I - \tau\}}{rn(\lambda+n)(2\lambda n+rn)} - \frac{X_0 \nu \sin\{n(\lambda-r)I - \tau\}}{rn(\lambda-n)(2\lambda n-rn)};$$

$$X_{5,r} = -\frac{X_0 \{a_{1,0} - (\lambda+n)^2\} \cos\{n(\lambda+r)I - \tau\}}{2rn(\lambda+n)^2(2\lambda n+rn)} \\ + \frac{X_0 \{a_{1,0} - (\lambda-n)^2\} \cos\{n(\lambda-r)I - \tau\}}{2rn(\lambda-n)^2(2\lambda n-rn)}.$$

Terms involving argument $\odot_{5,\lambda}$:

$$c_{5,\lambda}=0, \quad a_{5,\lambda}=0.$$

$$A_{5,\lambda} = \frac{1}{6} \frac{X_0 v \sin(2\lambda n I - \tau)}{\lambda^3 n^3} ;$$

$$X_{5,\lambda} = -X_0 (a_{1,0} - 4\lambda^2 n^2) \cos(2\lambda n I - \tau) \\ \frac{24\lambda^3 n^3}{\lambda^3 n^3}$$

[here in addition $\cos\tau=0$, in order to avoid the existence of I occurring explicitly in $A_{5,\lambda}$].

Terms involving argument $\odot_{5,2\lambda}$:

$$c_{5,2\lambda} = \frac{-v \sin 2\tau}{2\lambda^2 n^2} ; \quad a_{5,2\lambda} = \frac{-2v^2 \cos 2\tau}{\lambda^2 n^2}$$

$$A_{5,2\lambda} = \frac{X_0 v \sin(3\lambda n I - \tau)}{24\lambda^3 n^3} ;$$

$$X_{5,2\lambda} = \frac{X_0 (2\lambda^2 n^2 + v^2) \cos(3\lambda n I - \tau)}{36\lambda^3 n^3} + \frac{X_0 \cos 2\tau \cos(\lambda n I - \tau)}{2\lambda^2 n^2} \\ + \frac{X_0 (\lambda^2 n^2 - 2v^2) \sin 2\tau \sin(\lambda n I - \tau)}{4\lambda^3 n^3}$$

Terms involving products powers of \odot 's follow in a similar fashion.

If we summarise the parts specially required we find

$$\odot_{1,0} = (\lambda^2 n^2 - 4v^2) + a_{1,2\lambda} \odot_{1,2\lambda} + a_{2,2\lambda} \odot_{2,2\lambda} + a_{4,2\lambda} \odot_{4,2\lambda} \\ + a_{5,2\lambda} \odot_{5,2\lambda} + \dots$$

$$= (\lambda^2 n^2 - 4v^2) + \frac{1}{2} \cos 2\tau \odot_{1,2\lambda} - \frac{v}{\lambda n} \cos 2\tau \odot_{2,2\lambda} - \frac{v \cos 2\tau}{\lambda n} \odot_{4,2\lambda}$$

$$+ \frac{2v^2}{\lambda^3 n^3} \cos 2\tau \odot_{5,2\lambda} - \dots \quad [H]$$

and

$$c = c_{1,2\lambda} \odot_{1,2\lambda} + c_{2,2\lambda} \odot_{2,2\lambda} + c_{4,2\lambda} \odot_{4,2\lambda} + c_{5,2\lambda} \odot_{5,2\lambda} + \\ = \frac{1}{4} \frac{\sin 2\tau}{\lambda n} \odot_{1,2\lambda} - \frac{\nu \sin 2\tau}{2\lambda^2 n^2} \odot_{2,2\lambda} - \frac{\nu \sin 2\tau}{2\lambda^3 n^3} \odot_{4,2\lambda} - \frac{\nu \sin 2\tau}{2\lambda^3 n^3} \odot_{5,2\lambda} \quad [K],$$

where, as already stated

$$a_{1,0} = \lambda^2 n^2 - 4\nu^2 \text{ and } \cos \tau = 0.$$

It is necessary to examine the expressions just obtained in order to see whether the complete integral of the equations [B] has been obtained.

The integer λ is determined so as *most nearly* to satisfy the relation

$$\odot_{1,0} = \lambda^2 n^2 - 4\nu^2,$$

wherein everything excepting λ is known.

The negative value of λ will also satisfy the above relation.

Since $\cos \tau = 0$ always, altogether there are *four* distinct values of τ obtainable viz., $\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$. Each of these will give a distinct value

of c on substituting in [K] and different values for A and X . But fortunately, for our case c is always zero. Hence altogether we get *four* distinct solutions for A and X and these when multiplied by arbitrary constants will give the complete primitive of equations [B]. Such solutions as it is clear, will not contain the exponential factor.

Hence the solutions for ρ and σ are periodic functions and not pseudo-periodic as contemplated *a priori*.

(b) The Particular Integral.

We have now to determine the particular integral of equations [5] and [6] of Section III. We shall assume only one general term on the right-hand side and take the complete integral as the sum of a series of the corresponding solutions. The equations may therefore be written

$$\rho'' - 2\nu\rho' + \rho \sum_{r=0}^{\infty} \odot_{1,r} \cos rnI + \sigma \sum_{r=1}^{\infty} \odot_{2,r} \sin rnI = \frac{1}{2} \odot_{3,n} e^{imnI} \quad \dots [B'] \\ \sigma'' + 2\nu\rho' + \rho \sum_{r=1}^{\infty} \odot_{4,r} \sin rnI + \sigma \sum_{r=1}^{\infty} \odot_{5,r} \cos rnI = 0$$

Assume

$$\rho = e^{mnI} A,$$

$$\sigma = e^{mnI} X,$$

where A and X are as before functions of I .

Substituting these values of ρ and σ in [B'] we find

$$\begin{aligned} (A'' + 2mnA' - m^2 n^2 A) - 2\nu(X' + mnX) + A \sum_{r=0} \odot_{1,r} \cos rI \\ + X \sum_{r=1} \odot_{2,r} \sin rI = \frac{1}{2} \odot_{3,m} \quad \dots \quad [B' \cdot 1] \end{aligned}$$

$$\begin{aligned} (X'' + 2mnX' - m^2 n^2 X) + 2\nu(A' + mnA) + A \sum_{r=1} \odot_{1,r} \sin rI \\ + X \sum_{r=1} \odot_{3,r} \cos rI = 0 \quad \dots \quad [B' \cdot 2] \end{aligned}$$

Put

$$A = A_0 + \sum A_{r,s} \odot_{r,s} + \sum \sum B_{r,s,p,q} \odot_{r,s} \odot_{p,q} + \dots,$$

$$X = X_0 + \sum X_{r,s} \odot_{r,s} + \sum \sum Y_{r,s,p,q} \odot_{r,s} \odot_{p,q} + \dots,$$

in which $\odot_{1,0}$ is wanting. A_0, X_0 are constants, other coefficients of \odot 's are functions of I . Substitute these values of A and X in equations [B'·1] and [B'·2] and equate to zero the terms involving no \odot except $\odot_{1,0}$. We have then

$$\left. \begin{aligned} -m^2 n^2 A_0 - 2\nu mn X_0 + A_0 \odot_{1,0} &= \frac{1}{2} \odot_{3,m} \\ -m^2 n^2 X_0 + 2\nu mn A_0 &= 0 \end{aligned} \right\} \quad [B' \cdot 3]$$

whence

$$A_0 = \frac{1}{4} \odot_{3,m} \div (\odot_{1,0} - m^2 n^2 + \frac{1}{4} \nu^2)$$

$$X_0 = \nu \odot_{3,m} \div mn (\odot_{1,0} - m^2 n^2 + \frac{1}{4} \nu^2).$$

Equations [B'·1], [B'·2] can be fully written thus

$$\begin{aligned} (\sum A''_{r,s} \odot_{r,s} + \dots) + 2mn(\sum A'_{r,s} \odot_{r,s} + \dots) \\ - m^2 n^2 (A_0 + \sum A_{r,s} \odot_{r,s} + \dots) \end{aligned}$$

$$\begin{aligned}
& -2\nu\{(\sum X'_{r,s} \odot_{r,s} + \dots) + \nu mn(X_0 + \sum X_{r,s} \odot_{r,s} + \dots)\} \\
& + (A_0 + \sum A_{r,s} \odot_{r,s} + \dots) \sum_0 \odot_{1,r} \cos rnI \\
& + (X_0 + \sum X_{r,s} \odot_{r,s} + \dots) \sum_1 \odot_{2,r} \sin rnI = \frac{1}{2} \odot_{3,m} \quad [C' \cdot 1]
\end{aligned}$$

$$\begin{aligned}
& (\sum X''_{r,s} \odot_{r,s} + \dots) + 2mn(\sum X'_{r,s} \odot_{r,s} + \dots) \\
& - m^2 n^2 (X_0 + \sum X_{r,s} \odot_{r,s} + \dots) + 2\nu\{(\sum A'_{r,s} \odot_{r,s} + \dots) \\
& + mn(A_0 + \sum A_{r,s} \odot_{r,s} + \dots)\} + (A_0 + \sum A_{r,s} \odot_{r,s} + \dots) \\
& \sum_1^\infty \odot_{1,r} \sin rnI + (X_0 + \sum X_{r,s} \odot_{r,s} + \dots) \sum_1^\infty \odot_{3,r} \cos rnI = 0 \quad [D' \cdot 1]
\end{aligned}$$

Equate the coefficients of $\odot_{1,r}$ to zero :—

$$\begin{aligned}
A''_{1,r} + 2mnA'_{1,r} - m^2 n^2 A_{1,r} - 2\nu X'_{1,r} - 2\nu mn X_{1,r} \\
+ A_{1,r} \odot_{1,0} + A_0 \cos rnI = 0 \quad \dots [C' \cdot 2]
\end{aligned}$$

$$X''_{1,r} + 2mnX'_{1,r} - m^2 n^2 X_{1,r} + 2\nu A'_{1,r} + 2\nu mn A_{1,r} = 0 \quad \dots [D' \cdot 2]$$

In the equations [C'·2], [D'·2] first put e^{rnI} then e^{-rnI} for $\cos rnI$. Finally, the complete solutions of them will be obtained by adding up and halving the results thus found out.

On solving we get

$$\begin{aligned}
A_{1,r} &= \frac{1}{2} A_0 e^{rnI} \div [(mn+rn)^2 - \odot_{1,0} - 4\nu^2] \\
&+ \frac{1}{2} A_0 e^{-rnI} \div [(mn-rn)^2 - \odot_{1,0} - 4\nu^2] \\
X_{1,r} &= \nu A_0 e^{rnI} \div [(mn+rn)\{(mn+rn)^2 - \odot_{1,0} - 4\nu^2\}] \\
&+ \nu A_0 e^{-rnI} \div [(mn-rn)\{(mn-rn)^2 - \odot_{1,0} - 4\nu^2\}].
\end{aligned}$$

Now A_0 involves $\odot_{3,m}$, therefore $A_{1,r} \odot_{1,r}$, $X_{1,r} \odot_{1,r}$ each includes $\odot_{3,m} \odot_{1,r}$ as a factor. Hence these terms are negligible in comparison to A_0 and X_0 .

$$\text{Hence} \quad \rho = e^{imnI} A_0,$$

$$\sigma = e^{imnI} X_0,$$

where A_0, X_0 are given above. Now put $\rho = e^{-imnI} A_0, \sigma = e^{-imnI} X_0$ and we get the same values for A_0 and X_0 .

$$\text{Next put} \quad \frac{r}{2i} \odot_{n,m'} e^{imnI}$$

in the right-hand member of the second equation in $[B']$, and zero in the right-hand member of the first equation in $[B']$. We get the equations

$$\begin{aligned} \rho'' - 2\nu\rho' + \rho \sum_{r=0}^{\infty} \odot_{1,r} \cos rnI + \sigma \sum_{r=1}^{\infty} \odot_{2,r} \sin rnI &= 0 \\ \sigma'' + 2\nu\sigma' + \sigma \sum_{r=0}^{\infty} \odot_{3,r} \sin rnI + \rho \sum_{r=1}^{\infty} \odot_{4,r} \cos rnI &= \frac{1}{2i} \odot_{6,m} e^{imnI} \end{aligned} \quad \begin{aligned} & \\ & \quad [B''] \end{aligned}$$

Proceeding in the way mapped out as *ante*, we find

$$A_0 = -\nu mn \odot_{6,m} \div [m^2 n^2 (\odot_{1,0} - m^2 n^2 + 4\nu^2)],$$

$$X_0 = \frac{1}{2i} \odot_{6,m} (\odot_{1,0} - m^2 n^2) \div m^2 n^2 [\odot_{1,0} - m^2 n^2 + 4\nu^2].$$

As before all other terms in A, X are negligible, so we need not calculate them. For the complete solution we should calculate the corresponding terms for

$$-\frac{1}{2i} e^{-imnI}$$

which are easily obtained from those involving

$$1_{-imnI}$$

Thus

$$A_0 = -\nu mn \odot_{6,m} \div [m^2 n^2 (\odot_{1,0} - m^2 n^2 + 4\nu^2)],$$

$$X_0 = -\frac{1}{2i} \odot_{6,m} (\odot_{1,0} - m^2 n^2) \div [m^2 n^2 (\odot_{1,0} - m^2 n^2 + 4\nu^2)].$$

Hence

$$\rho = \sum_{m=0}^{\infty} [mn \odot_{s,m} - 2v \odot_{s,m}] \cos mnI + mn [\odot_{1,0} - m^2 n^2 + 4v^2],$$

$$\sigma = \sum_{m=0}^{\infty} [-2v \odot_{s,m} + mn \odot_{s,m} (\odot_{1,0} - m^2 n^2)] \sin mnI \\ \div mn [\odot_{1,0} - m^2 n^2 + 4v^2].$$

V

SUMMARY AND CONCLUSION.

The results of the above analysis may be thus briefly summarised :—

The perturbed orbit may be represented by

$$r = b(1 + \rho)$$

$$\theta = (\omega' - \omega)t + a + \sigma$$

where b = radius of the unperturbed circular orbit,

ω' = angular velocity of rotation of the outer electron,

ω = inner.....,

a = any arbitrary epoch,

ρ and σ are elements of perturbation. The above analysis shows that they are both periodic functions of $(\omega' - \omega)t$. The method adopted is that due to Goldsbrough who introduced a modification of the procedure in the theory of lunar perturbations first initiated by Hill and developed his results on the lines mapped out by Whittaker, Young and others.

In atomic problems, the interest does not lie in calculating the exact position of the satellite at different times as in the case of lunar motion. The problem is to quantize the orbits and to find out if from such quantized orbits, the energy can be calculated as a function of n and k , the total quantum number and the azimuthal quantum number respectively; and then to verify this energy with the spectral terms mp , md etc.

Hitherto, the quantization has been confined to very simple orbits—such as circular orbits by Bohr and elliptic orbits by Sommerfeld. Epstein¹ discussed the case of orbits subjected to the perturbations due to a uniform field and gave an explanation of the Stark effect; but Nicholson² finds that the method is not mathematically sound.

Hicks³ has recently raised an important objection to Sommerfeld's principle that in all mp and md orbits, $\int p_\phi dq_\phi = 2h$ and $3h$ respectively.

The above discussion shows that the handling of the general problem is much difficult than can be imagined. I have not yet succeeded in quantizing perturbed orbits, and therefore cannot say how far these investigations will support Sommerfeld's general theorem. This is in the course of my investigation.

A glance at the values of the several constants A_{\dots}, X_{\dots} , shows that the perturbed motion constitutes an ensemble of discrete harmonic oscillations having different frequencies. So far as the radial perturbed element ρ is concerned, it is easy to see, we must have a *range of vibrations* within the maximum and minimum. Under such circumstances, at any rate, we must expect that the perturbed system will not possess any sharply separated *stationary states*. The compound motion has rigorously a two-fold periodic character,—one, round the kernel in a closed periodic orbit for the unperturbed system *i.e.*, neglecting the *zusatzfeld*, two, *librations*—both radial and azimuthal—of the electron about the position it would have occupied at any instant for the unperturbed system, due to the quota of perturbing forces subjected to it by the *zusatzfeld* calculated in Sec. II.

So corresponding to a single stationary state in the unperturbed system there exists a multiple of slowly varied stationary states in the perturbed system, possessing a pronounced cycle; of course, the resultant frequency of the group of perturbation oscillations must be vanishingly small as compared with the time of revolution of the electron in the undisturbed state. But whether or not the motion is what is technically called conditionally periodic is difficult to judge *a priori*.

Bohr has laid down⁴ that for a transition between two of the states corresponding to the perturbed system a radiation is emitted “whose frequency stands in the same relationship to the periodic course of the variations in the orbit, as the spectrum of a simple periodic system does

¹ Sommerfeld, ‘*Atombau und Spektrallinien*’, Third ed., pp. 339-51.

² ‘*Phil. Mag.*’, July, 1922.

³ ‘*Phil. Mag.*’, Aug., 1922.

⁴ *Theory of spectra and Atomic constitution*, p. 39.

to its motion in the stationary states." Any more, quantization is possible by exhibiting a new phase of the adiabatic hypothesis first propounded by Ehrenfest¹, or what is strictly called the principle of "mechanical transformability" of stationary states. In that case, however, there is, an *a priori* probability of getting an almost identical series formula as obtained by Sommerfeld. Nevertheless, it is undesirable at this stage to try to incorporate an analysis and posit a principle having a feature somewhat foreign to what has been set forth hereto. This is deferred to a future occasion.

¹ Proc. Acad. Amsterdam, XVI, p. 591 (1914), *Phy. Zeitschr.* XV, p. 657 (1914).
Ann. d. Phys. LI, p. 327 (1916), *Phil. Mag.* XXXIII, p. 500 (1917).

ALGEBRA OF POLYNOMIALS

By

NRIPENDRANATH GHOSH

Chapter II

Expansions

12. The problem of expanding a given explicit function of a polynomial or a number of polynomials (and their derivatives) admits of an elegant treatment by means of the theorems established in the preceding chapter. The expansions obtained are of highly general character and cases may occur where these expansions fail to be consistent when numerical values are substituted for the variables involved. We shall not attempt to enquire into the validity of such expansions, but on the other hand, assume those conditions to be existing under which the expansions are arithmetically intelligible.

13. Let then $\phi(u_n)$, in Art 4, be expanded in a series of ascending powers of z of the form

$$A_0 + A_1 z + A_2 z^2 + \cdots + A_r z^r + \cdots$$

then since

$$\frac{d}{dz} \phi(u_n) = \Delta_{n0} \phi(u_n).$$

we must have

$$\frac{d}{dz} (A_0 + A_1 z + A_2 z^2 + \cdots) = \Delta_{n0} (A_0 + A_1 z + A_2 z^2 + \cdots).$$

$$A_1 + 2A_2 z + 3A_3 z^2 + \cdots = \Delta_{n0} A_0 + \Delta_{n0} A_1 z + \Delta_{n0} A_2 z^2 + \cdots;$$

whence comparing coefficients,

$$\Lambda_1 = \Delta_{a_0} \Lambda_0,$$

$$2\Lambda_2 = \Delta_{a_0} \Lambda_1,$$

$$3\Lambda_3 = \Delta_{a_0} \Lambda_2,$$

$$(r+1)\Lambda_{r+1} = \Delta_{a_0} \Lambda_r,$$

or when reduced

$$\Lambda_1 = \Delta_{a_0} \Lambda_0$$

$$\Lambda_2 = \frac{\Delta_{a_0}^2}{2} \Lambda_0,$$

$$\Lambda_3 = \frac{\Delta_{a_0}^3}{6} \Lambda_0,$$

$$\Lambda_{r+1} = \frac{\Delta_{a_0}^{r+1}}{r+1} \Lambda_0,$$

where Λ_0 is evidently equal to $\phi(u_0)$.

Thus by successive application of the operator Δ_{a_0} we have a means of calculating all the coefficients in the expansion of $\phi(u_a)$.

14. It can be inferred from the following typical calculations that any coefficient Λ_r (in above) is a linear homogeneous function of $\phi'(a_0), \phi''(a_0), \dots, \phi^{(r)}(a_0)$ only, the coefficient of any derivative $\phi^{(t)}(a_0)$ in Λ_r being a rational and integral function of degree t and weight r ($t \geq r$) involving $a_1, a_2, a_3, \dots, a_r$ only of the coefficients of u_a .

We have

$$A_1 = a_1 \phi'(a_0),$$

$$2 A_2 = a_1^2 \phi''(a_0) + 2a_2 \phi'(a_0),$$

$$3 A_3 = a_1^3 \phi'''(a_0) + 6a_1 a_2 \phi''(a_0) + 6a_3 \phi'(a_0),$$

$$4 A_4 = a_1^4 \phi^{(4)}(a_0) + 12a_1^2 a_2 \phi'''(a_0) + (12a_2^2 + 24a_1 a_3) \phi''(a_0) \\ + 24a_4 \phi'(a_0),$$

$$5 A_5 = a_1^5 \phi^{(5)}(a_0) + 20a_1^3 a_2 \phi^{(4)}(a_0) + (60a_1^2 a_3 + 60a_1 a_2^2) \phi'''(a_0) \\ + (120a_1 a_4 + 120a_2 a_3) \phi''(a_0) + 120a_5 \phi'(a_0).$$

The coefficients A 's in the expansion of $\phi(u_a)$ are connected by means of the operator Δ_{u_0} . This is, however, not the only connection existing among those coefficients. There are others and we proceed to find them.

15. Differential relations among the coefficients in the expansion of $\phi(u_a)$: —

We have

$$\frac{\partial}{\partial a_i} \phi(u_a) = \frac{\partial \phi}{\partial u_a} \cdot \frac{\partial u_a}{\partial a_i} = \frac{\partial \phi}{\partial u_a} \cdot z^i = z^i \frac{\partial}{\partial a_0} \phi(u_a);$$

and this holds true for all values $0, 1, 2, 3, \dots, n$ of

Since $\phi(u_a)$ is expanded in the form

$$A_0 + A_1 z + A_2 z^2 + \dots + A_n z^n,$$

we must have by the above identity

$$z^i \frac{\partial}{\partial a_i} (A_0 + A_1 z + A_2 z^2 + \dots) = z^i \frac{\partial}{\partial a_0} (A_0 + A_1 z + A_2 z^2 + \dots);$$

whence comparing coefficients

$$\frac{\partial A_0}{\partial a_r} = \frac{\partial A_1}{\partial a_r} = \frac{\partial A_2}{\partial a_r} = \cdots = \frac{\partial A_{r-1}}{\partial a_r} = 0,$$

$$\frac{\partial A_r}{\partial a_r} = \frac{\partial A_0}{\partial a_0},$$

$$\frac{\partial A_{r+1}}{\partial a_r} = \frac{\partial A_1}{\partial a_0},$$

$$\frac{\partial A_{r+2}}{\partial a_r} = \frac{\partial A_2}{\partial a_0},$$

and so on; where r may have any of the values $0, 1, 2, 3, \dots, n$.

16. These differential relations simplify the process of operation by Δ_{a_0} upon the coefficients A 's. Let us take from Art 13, the equation

$$(r+1)A_{r+1} = \Delta_{a_0} A_r,$$

i.e.,

$$\begin{aligned} (r+1)A_{r+1} &= \left(a_1 \frac{\partial}{\partial a_0} + 2a_2 \frac{\partial}{\partial a_1} + 3a_3 \frac{\partial}{\partial a_2} + \cdots + na_n \frac{\partial}{\partial a_{n-1}} \right) A_r \\ &= a_1 \frac{\partial A_r}{\partial a_0} + 2a_2 \frac{\partial A_r}{\partial a_1} + 3a_3 \frac{\partial A_r}{\partial a_2} \\ &\quad + \cdots + (r+1)a_{r+1} \frac{\partial A_r}{\partial a_r}, \quad (\text{if } r < n) \\ &= a_1 \frac{\partial A_r}{\partial a_0} + 2a_2 \frac{\partial A_{r-1}}{\partial a_0} + 3a_3 \frac{\partial A_{r-2}}{\partial a_0} \\ &\quad + \cdots + (r+1)a_{r+1} \frac{\partial A_0}{\partial a_0}, \quad (\text{by Art 15}) \\ &= \frac{\partial}{\partial a_0} (a_1 A_r + 2a_2 A_{r-1} + 3a_3 A_{r-2} + \cdots + (r+1)a_{r+1} A_0). \end{aligned}$$

The above also holds good if $r =$ or $> n$.

17. By means of the identity in Art 6, we get further relations among the coefficients A 's,

Since

$$\frac{1}{z} \phi(u_n) = \left(\Delta_{n2} + na_n z \frac{\partial}{\partial a_n} \right) \phi(u_n),$$

we must have

$$z \frac{d}{dz} (A_0 + A_1 z + A_2 z^2 + \dots) = \Delta_{n2} (A_0 + A_1 z + A_2 z^2 + \dots)$$

$$+ na_n z \frac{\partial}{\partial a_n} (A_0 + A_1 z + A_2 z^2 + \dots),$$

$$\text{or} \quad z^2 (A_1 + 2A_2 z + 3A_3 z^2 + \dots) = \Delta_{n2} (A_0 + A_1 z + A_2 z^2 + \dots)$$

$$+ na_n z^{n+1} \frac{\partial}{\partial a_n} (A_0 + A_1 z + A_2 z^2 + \dots) : \quad (\text{by Art 15})$$

Whence comparing coefficients

$$\Delta_{n2} A_0 = 0,$$

$$\Delta_{n2} A_1 = 0,$$

$$\Delta_{n2} A_2 = A_1,$$

$$\Delta_{n2} A_3 = 2A_2,$$

$$\dots \quad \dots$$

$$\Delta_{n2} A_n = (n-1)A_{n-1},$$

$$\Delta_{n2} A_{n+1} + na_n \frac{\partial A_0}{\partial a_n} = nA_n.$$

$$\left\{ \begin{array}{l} \Delta_{n2} A_{n+2} + na_n \frac{\partial A_1}{\partial a_n} = (n+1)A_{n+1}, \\ \Delta_{n2} A_{n+3} + na_n \frac{\partial A_2}{\partial a_n} = (n+2)A_{n+2}, \\ \dots \end{array} \right.$$

and so on.

These relations may be regarded as reciprocal to those in Art 13.

18. Allied expansions :—

There are other allied forms in which $\phi(u_n)$ may be expanded. The calculation of the coefficients in these expansions may be made to depend on the fundamental one in Art 13. The forms of these allied expansions are given below :—

$$(1) \quad \phi(a_0 + a_1 z) + A'_2 z^2 + A'_3 z^3 + A'_4 z^4 + \dots$$

$$(2) \quad \phi(a_0 + a_1 z + a_2 z^2) + A''_3 z^3 + A''_4 z^4 + \dots$$

$$(3) \quad \phi(a_0 + a_1 z + a_2 z^2 + a_3 z^3) + A'''_4 z^4 + \dots$$

and so on.

. Let us find A'_r in the first of these allied forms. We observe that A'_r must be a part of A_r . To specify that part we notice that A'_r vanishes when

$$a_0 = a_1 = a_2 = \cdots = a_r = 0;$$

so that A'_r is the residue of A_r left by removing that part which is not equal to zero when

$$a_2 = a_3 = a_1 = \cdots = a_r = 0.$$

Similar remark applies to other allied forms.

19. Expansion of a function involving a number of polynomials

Let $\phi(u_a, u_b, u_c, \cdots)$, in Art 7, be expanded in the form

$$(A)_0 + (A)_1 z + (A)_2 z^2 + (A)_3 z^3 + \cdots,$$

then since

$$\phi(u_a, u_b, u_c, \cdots) = (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} + \cdots) \phi(u_a, u_b, u_c, \cdots)$$

we must have

$$\frac{d}{dz} \{ (A)_0 + (A)_1 z + (A)_2 z^2 + (A)_3 z^3 + \cdots \}$$

$$= (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} + \cdots) \{ (A)_0 + (A)_1 z + (A)_2 z^2 + (A)_3 z^3 + \cdots \},$$

or $(A)_1 + 2(A)_2 z + 3(A)_3 z^2 + \cdots$

$$= (\Delta_{a0} + \Delta_{b0} + \Delta_{c0} + \cdots) \{ (A)_0 + (A)_1 z + (A)_2 z^2 + \cdots \}.$$

Representing the compound operator $\Delta_{a0} + \Delta_{b0} + \Delta_{c0} + \cdots$ by $(\Delta)_0$ and comparing the coefficients of like powers of z we have

$$(A)_1 = (\Delta)_0 (A)_0,$$

$$2(A)_2 = (\Delta)_0 (A)_1,$$

$$3(A)_3 = (\Delta)_0 (A)_2,$$

$$(r+1)(A)_{r+1} = (\Delta)_0 (A)_r;$$

or when reduced

$$(\Lambda)_1 = (\Delta)_0 (\Lambda)_0,$$

$$(\Lambda)_2 = \left(\frac{(\Delta)_0^2}{2} \right) (\Lambda)_0,$$

$$(\Lambda)_3 = \left(\frac{(\Delta)_0^3}{3} \right) (\Lambda)_0,$$

$$(\Lambda)_{r+1} = \left(\frac{(\Delta)_0^{r+1}}{r+1} \right) (\Lambda)_0,$$

where $(\Lambda)_0$ is evidently equal to $\phi(u_a, u_b, u_c, \dots)$.

Thus by successive application of the operator $(\Delta)_0$ we have a means of calculating all the coefficients in the expansion of $\phi(u_a, u_b, u_c, \dots)$.

20. Differential relations among the coefficients in the expansion of $\phi(u_a, u_b, u_c, \dots)$:—

We have

$$\frac{\partial}{\partial u_r} \phi(u_a, u_b, u_c, \dots) = \frac{\partial \phi(u_a, u_b, u_c, \dots)}{\partial u_a} \cdot \frac{\partial u_a}{\partial u_r},$$

$$\frac{\partial \phi(u_a, u_b, u_c, \dots)}{\partial u_a} = \frac{\partial}{\partial u_a} \phi(u_a, u_b, u_c, \dots)$$

and this holds true for all values $a = 1, 2, 3, \dots, n$ of a . Similarly

$$\frac{\partial}{\partial b_q} \phi(u_a, u_b, u_c, \dots) = z^q \frac{\partial}{\partial b_a} \phi(u_a, u_b, u_c, \dots);$$

which holds true for all values $a = 1, 2, 3, \dots, m$ of q ;

$$\frac{\partial}{\partial c_p} \phi(u_a, u_b, u_c, \dots) = z^p \frac{\partial}{\partial c_a} \phi(u_a, u_b, u_c, \dots);$$

which holds true for all values $a = 1, 2, 3, \dots, l$ of p ; and so

Referring to Art 15, the differential relations among the coefficients $(A)_r$ may be obtained with regard to each of the variables a 's, b 's, c 's... from the identities above.

By means of these differential relations $(r+1)(A)_{r+1}$ may be expressed in the form

$$\begin{aligned} & \frac{\partial}{\partial a_0} \{a_1(A)_r + 2a_2(A)_{r-1} + 3a_3(A)_{r-2} + \dots + (r+1)a_{r+1}(A)_0\} \\ & + \frac{\partial}{\partial b_0} \{b_1(A)_r + 2b_2(A)_{r-1} + 3b_3(A)_{r-2} + \dots + (r+1)b_{r+1}(A)_0\} \\ & + \frac{\partial}{\partial c_0} \{c_1(A)_r + 2c_2(A)_{r-1} + 3c_3(A)_{r-2} + \dots + (r+1)c_{r+1}(A)_0\} \\ & + \dots \end{aligned}$$

21. When the relative magnitudes of $l, m, n \dots$ are given it is possible to obtain, by means of art 8, further relations among the coefficients (A) 's by proceeding exactly in the same way as in Art 17.

There is a set of allied forms in which $\phi(u_a, u_b, u_c \dots)$ may be expanded. The coefficients in each of these allied expansions may be deduced from those in the fundamental one.

22. Expansion of a function involving a polynomial and its derivatives :—

Let

$$\phi(u_a, u'_a, u''_a, \dots, u^{(r)}_a),$$

in Art 9, be expanded in the form

$$\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2 + \dots,$$

then since

$$\frac{d}{dz} \phi(u_a, u'_a, u''_a, \dots, u^{(r)}_a) = \Delta_{a0} \phi(u_a, u'_a, u''_a, \dots, u^{(r)}_a),$$

we must have

$$\begin{aligned} & \frac{d}{dz} (\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2 + \bar{A}_3 z^3 + \dots) \\ & = \Delta_{a0} (\bar{A}_0 + \bar{A}_1 z + \bar{A}_2 z^2 + \dots), \end{aligned}$$

$$\begin{aligned}\text{or} \quad \bar{A}_1 + 2\bar{A}_2 z + 3\bar{A}_3 z^2 + \dots \\ = \Delta_{a_0} \bar{A}_0 + \Delta_{a_0} \bar{A}_1 z + \Delta_{a_0} \bar{A}_2 z^2 + \dots ;\end{aligned}$$

whence comparing coefficients,

$$\bar{A}_1 = \Delta_{a_0} \bar{A}_0,$$

$$2\bar{A}_2 = \Delta_{a_0} \bar{A}_1,$$

$$3\bar{A}_3 = \Delta_{a_0} \bar{A}_2,$$

and so on, where \bar{A}_0 is evidently equal to

$$\phi(a_0, a_1, 2a_2, 3a_3, \dots | ra_r).$$

The coefficients \bar{A} 's are connected only by Δ_{a_0} . It has not yet been possible to find other connections existing among them.

23. Expansion of a transformed polynomial :—

Let $u_a(\psi t)$, the transformed polynomial of $u_a(z)$, in art 11, be expanded in a series of ascending powers of t of the form

$$a_0 + a_1 t + a_2 t^2 + \dots,$$

then since

$$\frac{1}{\psi'(t)} \frac{d}{dt} \{u_a(\psi t)\} = \Delta_{a_0} \{u_a(\psi t)\},$$

we must have

$$\begin{aligned}\frac{1}{\psi'(t)} \frac{d}{dt} (a_0 + a_1 t + a_2 t^2 + \dots) \\ = \Delta_{a_0} (a_0 + a_1 t + a_2 t^2 + \dots),\end{aligned}$$

or

$$\begin{aligned}a_1 + 2a_2 t + 3a_3 t^2 + \dots \\ = \psi'(t) \Delta_{a_0} (a_0 + a_1 t + a_2 t^2 + \dots) \\ = \Delta_{a_0} \psi'(t) \{a_0 + a_1 t + a_2 t^2 + \dots\}.\end{aligned}$$

$\psi'(t)$ being known from the given transformation $z = \psi(t)$ (it is usual to restrict $\psi(t)$ to rational integral functions alone) we can express the right-hand side of the above identity in a series of ascending powers of t . Now comparing coefficients of like powers of t the coefficients a 's may be obtained. a_0 is evidently equal to $u_a(\psi 0)$.

If $\psi(t)$ be a rational and integral function of k th degree in t , the transformation is one of the k th order. If, moreover, $\psi(0)=0$, the transformation is called a simple transformation of the k th order.

A function of the transformed polynomial may similarly be expanded.

24. Polynomials of degree infinite :—

When the degree n of the polynomial u_n increases without limit it becomes the polynomial v_∞ of degree infinite. For finite numerical values of the variables such a polynomial may have an infinite value and the polynomial is said to be divergent (for those values of the variables). Otherwise the polynomial is said to be convergent.

We may extend (with necessary changes) the theorems of the last chapter and those of the present one to include polynomials of degree infinite provided initially they are convergent.

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ON THE MOTION OF A VISCOUS LIQUID BETWEEN TWO NON-CONCENTRIC CIRCULAR CYLINDERS.

BY

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INTRODUCTION.

1. As far as I know, the problem of translation of two non-concentric infinite circular cylinders in a viscous liquid has not been investigated by any writer, though a solution of the analogous problem of rotation of two circular cylinders in a viscous liquid has been given in a recent issue ¹ of the Proceedings of the Royal Society, by Dr. G. B. Jeffery.

In the present paper, I have discussed the problem of translation of two parallel infinite circular cylinders in a viscous liquid. The solution is different in form according as one cylinder does or does not enclose the other. In the former case the problem can be solved in finite terms and we shall get the current function of the "initial motion"; while in the latter case the problem is in general insoluble, that is to say, except in special circumstances, "there is no steady motion which satisfies all the necessary conditions."

THE CURRENT-FUNCTION.

2. Let

$$x + iy = c \tan \frac{1}{2} (\xi + i\eta)$$

Then

$$x = c \frac{\sin \xi}{\cosh \eta + \cos \xi}, \quad y = \frac{c \sinh \eta}{\cosh \eta + \cos \xi}.$$

$$\begin{aligned} h^2 &= \left(\frac{d\xi}{dx} \right)^2 + \left(\frac{d\xi}{dy} \right)^2 \\ &= \left(\frac{d\eta}{dx} \right)^2 + \left(\frac{d\eta}{dy} \right)^2 \\ &= c^{-2} (\cosh \eta + \cos \xi)^2. \end{aligned}$$

¹ *The Rotation of two Circular Cylinders in a Viscous Fluid*, Proc. Roy. Soc., Vol. A. 101, No. A. 709, (1922). p. 169.

and

$$r^2 = c^2 (\cosh \eta - \cos \xi) / (\cosh \eta + \cos \xi)$$

The current-function satisfies the equation

$$\nabla^4 \psi = 0$$

To find a solution, let us write

$$\psi = H_1 \sin \xi + H_2 \sin \xi / (\cosh \eta + \cos \xi)$$

$$c^2 \nabla^2 \psi = \sin \xi \{ (H_1'' - H_1) (\cosh \eta + \cos \xi)^2 + H_2'' (\cosh \eta + \cos \xi) - 2H_2' \sinh \eta \}$$

Operating on $c^2 \nabla^2 \psi$ by δ^2 which stands for $\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}$,

$$c^2 \delta^2 \nabla^2 \psi = \sin \xi \{ (H_1'''' - H_1''') (\cosh \eta + \cos \xi)^2 + 4(H_1'''' - H_1'') \sinh \eta \times (\cosh \eta + \cos \xi) + (H_1'''' - H_1'') \left(2 \cosh 2\eta + 2 \cosh \eta \cos \xi - (\cosh \eta + \cos \xi)^2 - 6 (\cosh \eta + \cos \xi) \cos \xi + 2(1 - \cos^2 \xi) \right) + H_2'''' (\cosh \eta + \cos \xi) - 4H_2''' \cos \xi - 4H_2'' \cosh \eta \}$$

Equating to zero the coefficients of the several powers of $\cos \xi$, we obtain the following equations

$$H_1'''' - 10H_1''' + 9H_1'' = 0 \quad \dots \quad (1)$$

$$2(H_1'''' - H_1''') \cosh \eta + 4(H_1'''' - H_1'') \sinh \eta - 6(H_1'''' - H_1'') \cosh \eta + H_2'''' - 4H_2'' = 0 \quad (2)$$

$$(H_1'''' - H_1''') \cosh^2 \eta + 4(H_1'''' - H_1'') \sinh \eta \cosh \eta + 3(H_1'''' - H_1'') \cosh^2 \eta + H_2'''' \cosh \eta - 4H_2''' \cosh \eta = 0 \quad \dots \quad (3)$$

The third equation is not independent but follows directly from the first two.

Solving we get

$$H_1 = (A \cosh 3\eta + B \sinh 3\eta + C \cosh \eta + D \sinh \eta)$$

$$H_2 = (-\frac{1}{4} A \cosh 4\eta - \frac{1}{2} B \sinh 4\eta + E \cosh 2\eta + F \sinh 2\eta + G\eta + H)$$

Therefore the stream-function can be written in the form

$$\psi = (A \cosh 3\eta + B \sinh 3\eta + C \cosh \eta + D \sinh \eta) \sin \xi$$

$$+ \left(-\frac{1}{2} A \cosh 4\eta - \frac{1}{2} B \sinh 4\eta + E \cosh 2\eta + F \sinh 2\eta + G\eta + H \right)$$

$$\sin \xi \\ (\cosh \eta + \cos \xi)$$

Let the two cylinders be defined by constant values of η say $\eta = \alpha$, $\eta = \beta$. We may take α positive and greater than β , then β will be positive or negative according as the first cylinder does or does not enclose the second.

Let the outer cylinder be moved with velocity V_1 and the inner one with velocity V_2 parallel to the axis of Y .

If we write

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

then since at the surface of the cylinder

$$u=0, \quad v=V_1,$$

the boundary conditions become, when $\eta = \alpha$

$$\frac{\partial \psi}{\partial \xi} = V_1 c \left\{ \frac{\cos \xi}{(\cosh \alpha + \cos \xi)} + \frac{\sin^2 \xi}{(\cosh \alpha + \cos \xi)^2} \right\}$$

$$\frac{\partial \psi}{\partial \eta} = -V_1 c \left\{ \frac{\sin \xi \sinh \alpha}{(\cosh \alpha + \cos \xi)^2} \right\} \quad (a)$$

and when $\eta = \beta$

$$\frac{\partial \psi}{\partial \xi} = V_2 c \left\{ \frac{\cos \xi}{(\cosh \beta + \cos \xi)} + \frac{\sin^2 \xi}{(\cosh \beta + \cos \xi)^2} \right\}$$

$$\frac{\partial \psi}{\partial \eta} = -V_2 c \left\{ \frac{\sin \xi \sinh \beta}{(\cosh \beta + \cos \xi)^2} \right\} \quad (a')$$

But when $\eta=a$, we get from the expression for the stream function

$$\begin{aligned} \frac{\partial \psi}{\partial \xi} &= \{A \cosh 3a + B \sinh 3a + C \cosh a + D \sinh a\} \cos \xi \\ &+ \left\{ -\frac{1}{2} A \cosh 4a - \frac{1}{2} B \sinh 4a + E \cosh 2a + F \sinh 2a + Ga + H \right\} \\ &\times \left\{ \frac{\sin^2 \xi}{(\cosh a + \cos \xi)^2} + \frac{\cos \xi}{(\cosh a + \cos \xi)} \right\} \\ \frac{\partial \psi}{\partial \eta} &= \{3 A \sinh 3a + 3 B \cosh 3a + C \sinh a + D \cosh a\} \sin \xi \\ &+ \{-2 A \sinh 4a - 2B \cosh 4a + 2E \sinh 2a + 2F \cosh 2a + G\} \\ &\times \frac{\sin \xi}{(\cosh a + \cos \xi)} - \left\{ -\frac{1}{2} A \cosh 4a - \frac{1}{2} B \sinh 4a + E \cosh 2a \right. \\ &\left. + F \sinh 2a + Ga + H \right\} \frac{\sin \xi \sinh a}{(\cosh a + \cos \xi)^2} \quad (b) \end{aligned}$$

From (a) and (b) we get the following equations

$$A \cosh 3a + B \sinh 3a + C \cosh a + D \sinh a = 0 \quad \dots \quad (4)$$

$$\begin{aligned} &-\frac{1}{2} A \cosh 4a - \frac{1}{2} B \sinh 4a + E \cosh 2a + F \sinh 2a \\ &+ Ga + H = V_1 c \quad (5) \end{aligned}$$

$$3A \sinh 3a + 3B \cosh 3a + C \sinh a + D \cosh a = 0 \quad \dots \quad (6)$$

$$-2A \sinh 4a - 2B \cosh 4a + 2E \sinh 2a + 2F \cosh 2a + G = 0 \quad (7)$$

together with four precisely similar equations obtained from these by writing β and V_1 for a and V_1 .

Solving, we have,

$$A=B=C=D=0.$$

$$E = \frac{(V_1 - V_2)c}{2} \frac{(\cosh 2a - \cosh 2\beta)}{(a-\beta) \sinh 2(a-\beta) + 1 - \cosh 2(a-\beta)}$$

$$F = \frac{(V_1 - V_2)c}{(a-\beta) \sinh 2(a-\beta) + 1 - \cosh 2(a-\beta)} \frac{(\sinh 2\beta - \sinh 2a)}{}$$

$$G = (V_1 - V_2)c \frac{\sinh 2(a-\beta)}{(a-\beta) \sinh 2(a-\beta) + 1 - \cosh 2(a-\beta)}$$

$$H = - \left\{ \frac{-2(V_1 + V_2)c + 2(V_1 + V_2)c \cosh 2(a-\beta) + 4c\beta V_1}{4(a-\beta) \sinh 2(a-\beta) + 1 - 4 \cosh 2(a-\beta)} \right. \\ \left. - aV_2 \right\} \frac{\sinh 2(a-\beta)}{}$$

Substituting the values of the constants we can obtain the stream-function.

THE PRESSURE.

3. We know that $\mu \nabla^2 \psi$ and p are conjugate functions.

Now

$$\begin{aligned} c^2 \nabla^2 \psi &= 2E (\sin 2\xi \cosh 2\eta + 2 \sin \xi \cosh \eta) \\ &+ 2F (\sin 2\xi \sinh 2\eta + 2 \sin \xi \sinh \eta) - 2G \sin \xi \sinh \eta \end{aligned}$$

Therefore

$$\begin{aligned} p &= -\frac{\mu}{c^2} \{ 2E (\cos 2\xi \sinh 2\eta + 2 \cos \xi \sinh \eta) \\ &+ 2F (\cos 2\xi \cosh 2\eta + 2 \cos \xi \cosh \eta) - 2G \cos \xi \cosh \eta \} + \text{constant.} \end{aligned}$$

THE RESISTANCE.

4. The formulæ for the elongation of the shear are

$$e = -f = \frac{1}{2} \left(\frac{\partial h^2}{\partial \xi} \frac{\partial \psi}{\partial \eta} + \frac{\partial h^2}{\partial \eta} \frac{\partial \psi}{\partial \xi} \right) + \frac{h^2}{\partial \xi \partial \eta} \frac{\partial^2 \psi}{\partial \xi \partial \eta},$$

$$\gamma = h^2 \left(\frac{\partial^2 \psi}{\partial \eta^2} - \frac{\partial^2 \psi}{\partial \xi^2} \right) + \frac{\partial h^2}{\partial \eta} \frac{\partial \psi}{\partial \eta} - \frac{\partial h^2}{\partial \xi} \frac{\partial \psi}{\partial \xi}$$

(Ibbotson, *Elasticity*.)

Substituting, we get

$$e=f=0.$$

When $\eta=a$,

$$\gamma=c^{-2}(\cosh a+\cos \xi)(4E \cosh 2a+4F \sinh 2a) \sin \xi$$

and when $\eta=\beta$,

$$\gamma=c^{-2}(\cosh \beta+\cos \xi)(4E \cosh 2\beta+4F \sinh 2\beta) \sin \xi$$

The resistance acting on the outer cylinder is given by

$$R_1 = \int - \left(U \frac{dy}{ds} + p \frac{dc}{ds} \right) ds$$

where $U=\mu\gamma$, and the integration is taken round the circle.

$$\begin{aligned} R_1 &= \frac{\pi\mu}{c} \{ 2(E \sinh 4a + F \cosh 4a) - 4(E \sinh 2a \\ &\quad + F \cosh 2a) + 2F + 2G(\cosh 2a + 1) \} \\ &= 4\pi\mu(V_1 - V_2) \frac{\sinh 2(a-\beta)}{\{(a-\beta) \sinh 2(a-\beta) + 1 - \cosh 2(a-\beta)\}} \end{aligned}$$

Similarly the resistance acting on the inner cylinder is given by

$$R_2 = -4\pi\mu(V_1 - V_2) \frac{\sinh 2(a-\beta)}{\{(a-\beta) \sinh 2(a-\beta) + 1 - \cosh 2(a-\beta)\}}$$

The formulæ for the resistances R_1 and R_2 take very simple forms when we put $a=0$ and $V_1=0$.

We then have the solution for a cylinder moving in a viscous liquid bounded by an infinite rigid plane.

$$R_2 = - \frac{4\pi\mu V_2 \sinh 2\beta}{\{\beta \sinh 2\beta + 1 - \cosh 2\beta\}}$$

MOTION PARALLEL TO THE AXIS OF x .

5. Proceeding in an exactly similar way as in the former case the expression for the stream function is given by

$$\begin{aligned} \psi &= (A \cosh 2\eta + B \sinh 2\eta + C\eta) \\ &+ \left\{ -\frac{1}{2}A \cosh 3\eta - \frac{1}{2}B \sinh 3\eta + E \cosh \eta + F \sinh \eta \right. \\ &\quad \left. + G\eta \cosh \eta + H\eta \sinh \eta \right\} / (\cosh \eta + \cos \xi) \end{aligned}$$

the absolute constant being omitted as it contributes nothing to velocity.

An expression similar to this was obtained by Jeffery in the paper cited in a different method, but the boundary condition being different the solution will be entirely different.

Let us suppose that the outer cylinder is moved with velocity U_1 and inner cylinder with velocity U_2 , parallel to the axis of x .

The boundary conditions are, when $\eta=a$

$$\frac{\partial \psi}{\partial \xi} = -U_1 c \frac{\sinh a \sin \xi}{(\cosh a + \cos \xi)^2}$$

$$\frac{\partial \psi}{\partial \eta} = -U_1 c \left\{ \frac{\cosh a}{(\cosh a + \cos \xi)} - \frac{\sinh^2 a}{(\cosh a + \cos \xi)^2} \right\} \quad (e)$$

together with two similar conditions for the other cylinder where β and U_2 are written for a and U_1 . But from the expression for the stream-function we get, when $\eta=a$,

$$\frac{\partial \psi}{\partial \xi} = \left\{ -\frac{1}{2} A \cosh 3a - \frac{1}{2} B \sinh 3a + E \cosh a + F \sinh a \right.$$

$$\left. + Ga \cosh a + Ha \sinh a \right\} \frac{\sin \xi}{(\cosh a + \cos \xi)^2}$$

$$\frac{\partial \psi}{\partial \eta} = (2A \sinh 2a + 2B \cosh 2a + C)$$

$$+ \left\{ -\frac{3}{2} A \sinh 3a - \frac{3}{2} B \cosh 3a + E \sinh a + F \cosh a \right.$$

$$\left. + G (\cosh a + a \sinh a) + H (\sinh a + a \cosh a) \right\} / (\cosh a + \cos \xi)$$

$$- \left\{ -\frac{1}{2} A \cosh 3a - \frac{1}{2} B \sinh 3a + E \cosh a + F \sinh a \right.$$

$$\left. + Ga \cosh a + Ha \sinh a \right\} \frac{\sinh a}{(\cosh a + \cos \xi)^2}$$

The boundary conditions give the equations

$$-\frac{1}{2}A \cosh 3a - \frac{1}{2}B \sinh 3a + E \cosh a + F \sinh a \\ + Ga \cosh a + Ha \sinh a = -U_1 c \sinh a \quad \dots \quad (8)$$

$$2A \sinh 2a + 2B \cosh 2a + C = 0 \quad \dots \quad (9)$$

$$-\frac{3}{2}A \sinh 3a - \frac{3}{2}B \cosh 3a + E \sinh a + F \cosh a \\ + G(a \sinh a + \cosh a) + H(a \cosh a + \sinh a) = -U_1 c \cosh a \quad \dots \quad (10)$$

together with three similar equations corresponding to the inner cylinder.

We have thus six equations but seven unknown quantities. But we know that ψ is a single-valued function. It follows therefore, that the velocities and consequently the pressure is a single-valued function. We can calculate p , the pressure, by noting that $\mu \nabla^2 \psi$ and p are conjugate functions. In this way we find that p contains the many-valued term $2G\xi$, so that we must have

$$G=0 \quad (11)$$

Now

$$c^2 \nabla^2 \psi = A\{1 + 4 \cosh \eta \cos \xi + 2 \cosh 2\eta \cos 2\xi\} \\ + B\{2 \sinh 2\eta \cos 2\xi + 4 \sinh \eta \cos \xi\} + 2H \cosh \eta \cos \xi \\ + 2E + 2H$$

Therefore

$$p = -\frac{\mu}{c^2} [A\{2 \sinh 2\eta \sin 2\xi + 4 \sinh \eta \sin \xi\} \\ + B\{2 \cosh 2\eta \sin 2\xi + 4 \cosh \eta \sin \xi\} + 2H \sinh \eta \sin \xi]$$

and

$$\gamma = c^{-2} \{L(a) (\cosh a + \cos \xi) + U_1 c \sinh a \cosh a \\ + U_1 c \sinh a \cos \xi + 4 (A \cosh 2a + B \sinh 2a) (\cosh a + \cos \xi)^2\}$$

The resistance R_1 can be calculated from the expression

$$R_1 = \int \left(p \frac{dy}{ds} - U \frac{dx}{ds} \right) ds$$

where $U = \mu\gamma$ and the integration is taken round the circle.

$L(a)$ in the expression for γ stands for the quantity

$$-\frac{9}{2}(A \cosh 3a + B \sinh 3a) + E \cosh a + F \sinh a$$

$$+ H (2 \cosh a + a \sinh a)$$

$$R_1 = -\frac{2\mu\pi}{c} \{ L(a) \cosh a + U_1 c \sinh a \cosh a + 4 (A \cosh 2a + B \sinh 2a)$$

$$+ 4 \sinh a (A \sinh 3a + B \cosh 3a) - 2H \sinh^2 a \}$$

$$= -\frac{4\mu\pi}{c} H, \text{ after simplification.}$$

$$= 4\mu\pi(U_1 - U_2)X$$

$$\frac{\{\cosh 2a + \cosh 2\beta - 4 + 4 \cosh (2a - 2\beta) - \cosh (2a - 4\beta) - \cosh (4a - 2\beta)\}}{[(\beta - a)\{\cosh (4a - 2\beta) + \cosh (2a - 4\beta) - 4 \cosh (2a - 2\beta) + 4$$

$$- \cosh 2a - \cosh 2\beta\} + \{\sinh (4a - 2\beta) - \sinh (4a - 4\beta) + 2 \sinh 2(a - \beta)$$

$$- 3 \sinh 2a + 3 \sinh 2\beta + \sinh (2a - 4\beta)\}]$$

Similarly the force acting on the other cylinder is found to be

$$R_2 = \frac{4\mu\pi}{c} H$$

When $a=0$, $U_1=0$

$$R_1 = \frac{4\mu\pi U_2}{\beta},$$

a very simple expression.

TRANSLATION OF TWO CYLINDERS IN AN INFINITE VISCOUS LIQUID.

6. In such a case, the velocity of the liquid does not vanish at infinity. To illustrate this point let us consider the motion parallel to the axis of y . The orthogonal components of velocity are

$$-h \frac{\partial \psi}{\partial \xi} \text{ and } h \frac{\partial \psi}{\partial \eta}$$

At infinity $\eta=0$, $\xi=\pi$

$$-h \frac{\partial \psi}{\partial \xi} = -3c^{-1}(E+H)$$

and

$$h \frac{\partial \psi}{\partial \eta} = -2c^{-1}(E+H).$$

Hence the motion being finite at infinity is inconsistent with the general supposition that the liquid is at rest at infinity. Hence the motion is impossible.

We should hardly wonder at this result. For Stokes has pointed out that the motion of a viscous liquid due to the translation of a circular cylinder never attains to a steady state, and our present problem is similar to that of Stokes.

TRANSVERSE VIBRATIONS OF A THIN ROTATING ROD AND OF A ROTATING CIRCULAR RING.

By

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I. INTRODUCTORY.

1. When a body rotates about an axis with constant angular velocity and is in relative equilibrium, every point of the body may be considered as being acted on by a force which varies as the distance of the point from the axis of rotation. The discussion of the vibrations of elastic solids acted on by such body-forces generally involves equations which cannot be solved in finite terms or in any convergent infinite series. It is probably due to this cause that very few problems relating to the vibration of rotating bodies have hitherto been solved. But an indirect method has often been applied in such cases to obtain the frequencies of vibrations which are very approximate for all practical purposes. This approximate method is due to Lord Rayleigh and one very interesting problem has been dealt with by Prof. Lamb and Mr. R. V. Southwell.¹ They have investigated the transverse vibrations of a thin homogeneous circular disc rotating about its axis with constant angular velocity. They observe that "the problem has a practical bearing, as throwing light on the occasional failure of turbine discs," which is most probably due to the transverse vibrations of these discs, causing the blades which are fitted to them, to come in contact with the adjacent parts of the machine. This problem of the laminar wheel suggests the case of a wheel with straight spokes and a circular rim, which is by no means a less common thing in mechanical contrivances.

¹ "Vibrations of a Spinning Disc"—Proc. Roy. Soc., London. Ser. A., Vol. 99 (1921), pp. 272-280.

"On the Free Transverse Vibrations of Uniform Circular Disc clamped at its centre; and on the Effects of Rotation"—R. V. Southwell, Proc. Roy. Soc., London, Ser. A., Vol. 101 (1922), pp. 133-153.

2. It is clear that the discussion of the problem naturally resolves into two distinct parts, *viz.*, (1) the vibrations of the straight spokes, and (2) the vibrations of the circular rim. Both the spokes and the rim will be assumed to have small cross-sections, so that the effects of what is known as 'rotatory inertia' will be negligible. A spoke can vibrate transversally in two ways, either in the plane of the wheel or in a plane perpendicular to it. The mathematical solution is identical in the two cases. The rim may also vibrate in the same two ways; but the equations of motion are different, though it is known that the frequencies of the gravest modes of free vibration are very nearly the same.¹ When the spokes and the rim are taken as forming one body, the solutions become very complicated on account of the points of junction. In the work of the present paper, they are considered as separate bodies and independent solutions have been obtained for a thin rotating rod and a rotating circular ring.

II. THIN ROTATING ROD.

3. Suppose that a rod (AB) of length a is rotating about A with constant angular velocity ω . Since the rod is thin, we assume the stress-system to consist of a longitudinal tension (T_x) only. If A be taken as origin and the axis of x along AB, we have

$$\frac{\partial T_x}{\partial x} + \rho\omega^2 x = 0,$$

whence

$$T_x = \frac{1}{2}\rho\omega^2 (A - x^2)$$

4. Case A. Let the end B be free, so that $T_x = 0$ when $x = a$ and we have

$$T_x = \frac{1}{2}\rho\omega^2 (a^2 - x^2) \quad (1)$$

Case B. Let a mass m [*e.g.* (mass of the rim)/(number of spokes)] be attached to B, so that when $x = a$, we have

$$T_x = m\omega^2 a.$$

Hence, in this case

$$T_x = \frac{1}{2}\rho\omega^2 \left\{ a \left(a + \frac{2m}{\rho} \right) - x^2 \right\} \quad (2)$$

¹ Rayleigh Theory of Sound, Vol. I, Art. 192 a. Love, Elasticity, Chap. XXI, Art. 293.

5. Both the forms (1) and (2) may be included in the formula

$$T_x = \frac{1}{2} \rho \omega^2 (c^2 - x^2) \quad \dots \quad (3)$$

where

$$c^2 = a^2 \text{ or } a \left(a + \frac{2m}{\rho} \right),$$

according as the end B is free or carries a mass m .

When ω is very large and the flexural forces are negligible compared with the longitudinal tension, the equation of transverse vibration is

$$\rho a dx \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x} \left[T_x \cdot a \frac{\partial v}{\partial x} \right] dx,$$

where a is the small cross-section, and v , the lateral displacement of an element of the bar at a distance x from the origin.

Substituting from (3) the value of T_x , we have

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{2} \omega^2 \frac{\partial}{\partial x} \left[(c^2 - x^2) \frac{\partial v}{\partial x} \right]$$

Assuming the solution

$$v = f(x) \cos(p_1 t + \epsilon)$$

we have

$$(c^2 - x^2) \frac{\partial^2 f}{\partial x^2} - 2x \frac{\partial f}{\partial x} + b^2 f = 0, \quad (4)$$

where

$$b^2 = \frac{2p_1^2}{\omega^2} \quad (5)$$

To solve this, it will be convenient to assume a series in ascending powers of $\frac{x}{c}$ a quantity which is never greater than unity. Let us assume

$$f(x) = A_0 + A_1 \frac{x}{c} + A_2 \left(\frac{x}{c} \right)^2 + \dots + A_k \left(\frac{x}{c} \right)^k + \dots \quad (6)$$

Substituting this in (4), we get

$$(c^2 - x^2) \left[\dots + \frac{k(k-1)}{c^k} A_k x^{k-2} + \dots + \frac{(k+2)(k+1)}{c^{k+2}} A_{k+2} x^k + \dots \right] \\ - 2x \left[\dots + \frac{k}{c^k} A_k x^{k-1} + \dots \right] + b^2 \left[\dots + \frac{A_k}{c^k} x^k + \dots \right] = 0$$

Equating the coefficients of x^k to zero, we have

$$(k+2)(k+1)A_{k+2} = \{k(k+1) - b^2\} A_k.$$

Calculating the coefficients of (6) by this formula, we obtain

$$f(x) = A_0 S_0(x) + A_1 S_1(x),$$

where A_0 and A_1 are constants and $S_0(x)$ and $S_1(x)$ stand for the following series :

$$S_0(x) = 1 - \left[\frac{b^2}{2!} \left(\frac{x}{c} \right)^2 + \frac{b^2(3 \cdot 2 - b^2)}{4!} \left(\frac{x}{c} \right)^4 \right. \\ \left. + \frac{b^2(3 \cdot 2 - b^2)(5 \cdot 4 - b^2)}{6!} \left(\frac{x}{c} \right)^6 \right. \\ \left. + \dots + \frac{b^2(3 \cdot 2 - b^2) \dots \{(2x-1)(2x-2) - b^2\}}{(2x)!} \left(\frac{x}{c} \right)^{2x} + \dots \right] \\ S_1(x) = \frac{x}{c} + \frac{2 \cdot 1 - b^2}{3!} \left(\frac{x}{c} \right)^3 - \frac{(2 \cdot 1 - b^2)(4 \cdot 3 - b^2)}{5!} \left(\frac{x}{c} \right)^5 + \dots \\ + \frac{(2 \cdot 1 - b^2)(4 \cdot 3 - b^2) \dots \{2x(2x-1) - b^2\}}{(2n+1)!} \left(\frac{x}{c} \right)^{2x+1} + \dots$$

The complete solution is therefore

$$x = [A_0 S_0(x) + A_1 S_1(x)] \cos(p_1 t + \epsilon) \quad \dots \quad (7)$$

6. We have assumed the end A (i.e. $x=0$) to be fixed, so that we must have $v=0$ when $x=0$. This shews that we must put $A_0=0$, and the appropriate solution is

$$v = A_1 S_1(x) \cos(p_1 t + \epsilon) \quad \dots \quad (8)$$

The series $S_1(x)$ is convergent when $x < c$ but it is divergent when $x=c$ or when $x > c$. We have now to distinguish between the two cases indicated in Art. 4 above.

In case A., we have $x=c(=a)$ at the edge, the series $S_1(x)$ is divergent and the solution is meaningless unless the series consists

of a finite number of terms. Hence we see from the form of $S_1(x)$ that, in order that the series may terminate, b^2 must be of the form $2n(2n-1)$, where n is any positive integer. We therefore have

$$b^2 = 2n(2n-1)$$

or by (5),

$$p_1^2 = n(2n-1)\omega^2, \quad \dots \quad (9)$$

n being any positive integer.

In case B., we have (from Art. 5)

$$v = a \left(a + \frac{2m}{\rho} \right)$$

and x is always less than c . The series $S_1(x)$ is therefore always convergent. The condition of the end $x=a$, may be expressed by

$$\left[m \frac{\partial^2 v}{\partial t^2} \right]_{x=a} = \left[-\omega^2 a \cdot m \frac{\partial v}{\partial x} \right]_{x=a}$$

Substituting for v , this becomes

$$p_1^2 S_1(a) - \frac{\omega^2 a}{c} S_1'(a) = 0,$$

$$\text{or} \quad b^2 S_1(a) - \frac{2a}{c} S_1'(a) = 0 \quad (10)$$

which is an equation in p_1^2 .

7. When, on the other hand, the influence of rotation is small compared with the flexural forces, we know that, the rotatory inertia of the cross-section of the rod being neglected, the equation of motion is

$$\frac{\partial^2 v}{\partial t^2} + \frac{E k^2}{\rho} \frac{\partial^4 v}{\partial x^4} = 0$$

where k is the radius of gyration of the cross-section about a diameter perpendicular to the plane of vibration. If p_2 be the frequency, it is given by

$$p_2^2 = \frac{m^4 k^2}{4\pi^2 a^4} \frac{E}{\rho} \quad (11)$$

where m is given, in the case of a free-free bar, by $\cosh m c \cos m = 1$. and in the case of a clamped-free bar, by $\cosh m c \cos m = -1$.

8. When both the flexural and the centrifugal forces are taken into account, the equation of motion becomes

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{2}\omega^2 \frac{\partial}{\partial x} \left[(c^2 - x^2) \frac{\partial v}{\partial x} \right] - \frac{Ek^2}{\rho} \frac{\partial^4 v}{\partial x^4}.$$

If we assume

$$v = f(x) \cos (pt + \epsilon)$$

we have

$$\frac{2Ex^2}{\omega^2 \rho} \frac{\partial^4 f}{\partial x^4} - (c^2 - x^2) \frac{\partial^2 f}{\partial x^2} + 2x \frac{\partial f}{\partial x} - b^2 f = 0.$$

If a series analogous to (6) be substituted in this equation, the relation between the successive coefficients consists of three terms (*e.g.*, A_{k+4} , A_{k+2} , A_k) so that a general solution in finite terms or in a convergent infinite series is not easily obtainable.

We may, however, obtain approximate solutions by a method¹ indicated by Rayleigh.² According to this method, we may assume a given form for the displacement v , calculate the kinetic energy and equate this to the sum of the potential energies due to the angular motion and the flexural forces considered separately. The equation thus obtained yields the frequency of vibration.

We proceed to apply this method to the case A of Art. 4. The potential energy V of the centrifugal forces is given by

$$V = \frac{1}{2} \int a T_x \left(\frac{\partial v}{\partial x} \right)^2 dx,$$

where a = cross-section of the rod and

$$T_x = \frac{1}{2} \rho \omega^2 (a^2 - x^2)$$

The potential energy of the flexural forces is given by

$$V' = \frac{1}{2} \int Ek^2 a \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx$$

The kinetic energy is given by

$$T = \frac{1}{2} \rho \int \left(\frac{\partial v}{\partial t} \right)^2 a dx.$$

¹ This method has also been adopted by Prof. Lamb and Mr. R. V. Southwell in the papers cited.

² Theory of Sound, Vol. I, Chap. IV, Arts. 88 *et seq.*

Let us assume the form

$$v=f(x) \cos (pt+\epsilon).$$

Then

$$V=\frac{1}{2}\rho\omega^2a\int(a^2-x^2)\left(\frac{\partial f}{\partial x}\right)^2\cos^2(pt+\epsilon)dx$$

$$V'=\frac{1}{2}Ek^2a\int\left(\frac{\partial^2 f}{\partial x^2}\right)^2\cos^2(pt+\epsilon)dz$$

and

$$\begin{aligned} T &= \frac{1}{2}\rho a \int p^2 f^2 \sin^2(pt+\epsilon)dx \\ &= p^2 \cdot \frac{1}{2}\rho a \int f^2 \sin^2(pt+\epsilon)dx \end{aligned}$$

If V_f , V'_f and T_f denote the mean values of V and V' , and the expression

$$\frac{1}{2}\rho a \int f^2 \sin^2(pt+\epsilon)dx$$

respectively, we have

$$p^2 = \frac{V_f + V'_f}{T_f} \quad \dots \quad (12)$$

The closer the assumed function $f(x)$ agrees with the actual form of the vibrating bar, the more will the value of p^2 approach

$$(V_f + V'_f)/T_f.$$

Moreover, the frequency remains stationary for small deviations from the actual type. Hence, if p_1 and p_2 be the two values of the frequency, obtained from the equation (9) or (10) and (11) respectively, we have very approximately

$$p_1^2 = \frac{V_f}{T_f}, \quad p_2^2 = \frac{V'_f}{T_f}$$

and

$$p^2 = p_1^2 + p_2^2 \quad \dots \quad (13)$$

9. Assume as an example that

$$f(x) = A_1 \left\{ \frac{x}{a} + m \left(\frac{x}{a} \right)^3 \right\} \quad \dots \quad (14)$$

where m is a variable parameter whose value is to be determined from the fact that the value of the period given by equation (12) should be a minimum.¹ Let us now calculate the values of V_f , V'_f and T_f . Since the mean values of

$\cos^2(pt + e)$ or $\sin^2(pt + e)$ is $\frac{1}{2}$, we find

$$\begin{aligned} V_f &= \frac{1}{8} \rho \omega^2 a A_1 \int_0^a (a^2 - x^2) \left\{ \frac{1}{a} + \frac{3m}{a^3} x^2 \right\}^2 dx \\ &= \frac{1}{8} \frac{\rho \omega^2 a A_1}{a^6} \int_0^a \{ a^6 + (6m-1)a^4 x^2 \\ &\quad + (9m^2 - 6m)a^2 x^4 - 9m^2 x^6 \} dx \\ &= \frac{\rho \omega^2 a A_1}{4 \cdot 105} (27m^2 + 42m + 35) \end{aligned}$$

$$\begin{aligned} V'_f &= \frac{1}{4} E k^2 a A_1 \int_0^a \left(\frac{6mx}{a^3} \right)^2 dx \\ &= \frac{3 E k^2 a A_1}{a^3} m^2 \end{aligned}$$

$$\begin{aligned} T_f &= \frac{1}{4} \rho a A_1 \int_0^a \left\{ \frac{x}{a} + m \left(\frac{x}{a} \right)^3 \right\}^2 dx \\ &= \frac{\rho a A_1}{4 \cdot 105} (15m^2 + 42m + 35) \end{aligned}$$

¹ See remarks by Mr. R. V. Southwell in the paper "Vibrations of a Spinning Disc,"

As a partial verification of the above results, let us make $n=2$ in equation (9), so that

$$p_1^2 = 6\omega^2.$$

The corresponding value of b^2 is 12, the series $S_1(x)$ terminates at the second term and

$$S_1(x) = A_1 \left\{ \frac{x}{a} - \frac{5}{3} \left(\frac{x}{a} \right)^2 \right\},$$

so that

$$m = -\frac{2}{3}.$$

Substituting this value of m in the expressions for V_f and T_f found above, we see that

$$V_f = \frac{2A_1}{21} \rho \omega^2 a a$$

and

$$T_f = \frac{A_1}{63} \rho a a$$

so that

$$p_1^2 = \frac{V_f}{T_f} = 6\omega^2,$$

which is the same as that obtained from equation (9) by putting $n=2$.

To return to our general case, we have

$$V_f + V'_f = \frac{\rho \omega^2 a a}{420} (27m^2 + 42m + 35) + \frac{3Ek^2 a}{\sigma^2} m^2$$

and

$$p^2 = \frac{\frac{\rho \omega^2 a a}{420} (27m^2 + 42m + 35) + \frac{3Ek^2 a}{\sigma^2} m^2}{\frac{\rho a a}{420} (15m^2 + 42m + 35)}$$

If for brevity, we put

$$\left. \begin{aligned} A &= \frac{1}{15} \rho \omega^2 a + \frac{3Ek^2}{a^3} \\ B &= \frac{1}{15} \rho \omega^2 a, & C &= \frac{1}{15} \rho \omega^2 a \\ A' &= \frac{1}{15} \rho a, & B' &= \frac{1}{15} \rho a, & C' &= \frac{1}{15} \rho a \end{aligned} \right\}, \quad \dots \quad (15)$$

we get

$$p^2 = \frac{Am^2 + Bm + C}{A'm^2 + B'm + C'}$$

We have now to find m in order that the values of p^2 may be stationary. The corresponding values of m are given by

$$(AB' - A'B)m^2 - 2(AC' - A'C)m + BC' - BC = 0 \quad \dots \quad (16)$$

and the values of p^2 by

$$\begin{aligned} (A'C' - \frac{1}{15}B'^2)p^2 - (C'A + A'C - \frac{1}{15}BB')p^2 \\ + AC - \frac{1}{15}B^2 = 0 \quad \dots \quad (17) \end{aligned}$$

The values of m and p^2 may be calculated when the values of the constants (15) are known, and the true value of the frequency will be obtained, if the assumed form (14) is appropriate.

III. ROTATING CIRCULAR RING.

10. We assume that a circular ring of radius a and small cross-section, rotating in its plane with constant angular velocity ω , is vibrating transversally, the displacements being perpendicular to the plane of the ring. If p_1 and p_2 be the values of the frequency in the two extreme cases, viz., (1) when the flexural forces are negligible and (2) when the angular motion is negligible, then, according to our observations in Art. 8, we have very approximately

$$p^2 = p_1^2 + p_2^2.$$

It is known¹ that, when the rotatory inertia is neglected, the value of p_2^2 is given by

$$p_2^2 = \frac{E\pi c^4}{4ma^4} \frac{n^2(n^2 - 1)^2}{n^2 + 1 + \sigma} \quad \dots \quad (18)$$

where c is the radius of the cross-section, m the mass per unit length and n is any integer.

¹ Love, *Elasticity*, Art. 293 (b) or Michell, *Messenger of Mathematics*, XIX, 1889.

We proceed to find p_1 .

11. Taking the centre of the ring as origin and (a, θ) the polar co-ordinates of any point on the circumference, we have, assuming the stress-system to consist of a longitudinal tension only,

$$-\frac{1}{a}\theta\theta + \rho\omega^2 a = 0,$$

whence

$$T_\theta (= \theta\theta) = \rho\omega^2 a^2 \quad (19)$$

The equation of motion is accordingly

$$\rho a \frac{\partial^2 v}{\partial t^2} a d\theta = - \frac{\partial}{\partial \theta} \left[a T_\theta \frac{\partial v}{a \partial \theta} \right] d\theta$$

or

$$\frac{\partial^2 v}{\partial t^2} = \omega^2 \frac{\partial^2 v}{\partial \theta^2} \quad (20)$$

The solution of this equation is

$$v = A \cos(\mu\theta + \beta) \cos(p_1 t + \epsilon)$$

where

$$\mu^2 = \frac{p_1^2}{a^2}$$

(i) If the point $\theta=0$ of the ring is relatively fixed, we have

$$v = A \sin \mu\theta \cos(p_1 t + \epsilon).$$

Since, in this case, $v=0$ when $\theta=2s\pi$, s being any integer, we have

$$\sin 2\mu s\pi = 0,$$

so that $2\mu = \frac{k}{s}$, k and s being any integers and

$$p_1 = \frac{\omega}{2s} \quad \dots \quad (22)$$

(ii) If two diametrically opposite points, $\theta=0$ and $\theta=\pi$, are fixed, we must have

$$\sin \mu\pi=0$$

so that

$$\mu=s, \quad \text{any integer,}$$

and

$$p_1=sw$$

(iii) If the ends of a quadrant, $\theta=0$ and $\theta=\frac{\pi}{2}$, are fixed, we have

$$\sin \frac{\mu\pi}{2}=0$$

and

$$p_1=2s\omega,$$

where s is any integer.

(iv) Generally, if the ends of the arc, $\theta=0$ and $\theta=\frac{2\pi}{n}$, are fixed, then

$$\sin \mu \frac{2\pi}{n}=0$$

whence

$$p_1=\frac{1}{2}n s w,$$

s being any integer.

The solution (18) for p_2 refers to a complete ring. Hence the corresponding solution for p_1 may be taken from (22), and the period, when both the angular velocity and the flexural forces are taken into account, will then be given by the equation

$$p^2=p_1^2+p_2^2.$$

The results in (ii), (iii), (iv) give very simple relations between the angular velocities and periods of free transverse vibrations of thin flexible rotating arcs of any angle clamped at the extremities.

GEOMETRICAL REPRESENTATION OF EQUATIONS OF CONICS FOR COMPLEX VARIABLES

By

MANUJANATH GHATAK

Chapter I

§1. *The necessity for the introduction of four-dimensional space.*

The geometric representation of an analytic equation in x and y is ordinarily obtained by the admission of only real values for the variables x and y . The imaginary or complex values have no place there, and where two such equations have imaginary or complex solutions, we get no points in which the corresponding geometrical figures intersect. Hence we get the phenomenon in Conic Sections of a straight line and a conic sometimes intersecting and sometimes not intersecting, whereas the analytical equations always have solutions. In the latter case we say, to bring the geometrical phenomenon in line with analytical results, that the line intersects the conic in imaginary points. What is really the case is that there are no points common to the line and the conic.

The anomaly arises out of the fact that the roots of an equation with real coefficients give rise to numbers which are not always real. The Argand's diagram gives us a method of representing all these numbers in a plane, and the totality of these numbers covers up the entire two dimensional plane region. The real numbers as well as the purely imaginary numbers are but particular cases of complex numbers

Since for the adequate representation of a single complex variable X we require a plane or a space of two dimensions, the adequate representation of two complex variables X & Y would require two planes in a space of four dimensions, having a common point at the origin. The four coordinate axes will lie two and two in the two planes. In each plane there are an axis of reals and an axis of imaginaries which are at right angles to each other. All the coordinate axes may be at right angles to one another, but we may have sometimes to deal with oblique axes. It should be remembered, however, that the axis of

imaginaries is necessarily at right angles to the axis of reals, but the angles between the axes in X-plane and axes in Y-plane will not always be right angles. In the case of equations to the Conic Sections, if we admit of complex values for the variables and substitute $\alpha+i\beta$ for x and $\gamma+i\delta$ for y , a single relation connecting x and y will be equivalent to two relations connecting α, β, γ and δ , which are obtained by equating the real and imaginary parts separately to zero. Hence we obtain that the equations really represent *surfaces*, whose sections in the plane of reals are what we ordinarily consider to be their geometrical interpretation. In reality, therefore, the equations to the straight line and the conic represent something more than the line or the conic in the real plane. They represent surfaces, and if the line and the conic in the real plane have no points of intersection, and still we can find solutions to the *equations*, we conclude that the *surfaces* intersect in some points outside the plane of reals.

As an illustration, let us take a case where the solution for one of the variables is purely imaginary, and see whether a three-dimensional space will give a geometrical solution.

Let the equations be $x^2+y^2=25$ and $x=\pm c$. When $|c|<5$ the solutions are real and the circle and the straight line in the real plane intersect in two points.

When $|c|=5$, the straight line touches the circle, and when $|c|>5$, there are no real solutions and the straight line and the circle do not meet.

In the real-imaginary plane, on the other hand, the curves are the hyperbola $x^2-y^2=25$, and the straight line $x=\pm c$; and when $|c|>5$, the straight line and the hyperbola intersect in two points; when $|c|=5$ they touch, the point of contact being the same as in the previous case, and is the common point of the circle and the hyperbola.

In each case, the points of intersection are those in which the *surface* $x^2+y^2=25$ intersects the *surface* $x=\pm c$, the point of contact being the point where $x=\pm 5$ intersects $x^2+y^2=25$. Where the geometrical solutions were unavailable in the real plane, they were obtained in the real-imaginary plane.

Similar arguments apply with regard to the equations $x^2+y^2=25$ and $y=\pm c$, the alternative plane of solution being the imaginary-real plane. The equations $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $x=\pm c$ or $y=\pm d$ are also of the same nature. The critical value in the case of $x=\pm c$ is $|c|=a$, and the alternative plane of solutions is the real-imaginary plane; in

the case of $y = \pm d$ the critical value is $|y| = b$, and the alternative plane the imaginary-real plane. Similar considerations apply with regard to the equations $y^2 = 4ax$ and $x = c$, the manifestation of the surface in the real-imaginary plane is $y^2 + 4ax = 0$ and the critical value is $x = 0$.

In all these cases, we have deliberately chosen the equations in such a way that a three-dimensional space suffices to show all the points of intersection. This mode of representation has helped to show that the curve in the real plane is not the whole of the surface represented by the second degree equation, and that there are other planes where we also get curves of intersection. It is difficult, however, to get an accurate conception of a surface in four dimensions; we can study only its curve-sections, and imagine that the surface is made up of all these curves. We should remember, however, that not all planes give curves of section, and we shall have to choose our planes in such a way as to make this possible. We shall show later, that in the case at least of equations of first and second degree, a single infinity of planes may in all cases be obtained where we get curves of section, and that the totality of all these curves represents the entire surface.

§2. *The plane in four dimensions.*

1. *The most general equations. Solid of the first degree.*

When the equation in four dimensions is of the first degree, we might call it a *solid of the first degree*

The most general scheme of transformation of coordinates may be written.

$$\alpha' = a_1\alpha + b_1\beta + c_1\gamma + d_1\delta + e_1$$

$$\beta' = a_2\alpha + b_2\beta + c_2\gamma + d_2\delta + e_2$$

$$\gamma' = a_3\alpha + b_3\beta + c_3\gamma + d_3\delta + e_3$$

$$\delta' = a_4\alpha + b_4\beta + c_4\gamma + d_4\delta + e_4$$

and by its aid any equation of the first degree may be transformed into $\delta = 0$. We might, therefore, get a conception of a solid of the first degree from the equation $\delta = 0$ which embraces a three-dimensional Euclidean space. Any equation of the first degree would then be the analytical equivalent of the solid being given any desired positions in a four dimensional space.

We shall now prove that a plane in four-dimensional space is given by the intersection of any two equations of the first degree in four variables.

The scheme of transformation given above would give a corresponding system of new axes. By this transformation any two equations may be reduced to the form $\beta'=0, \delta'=0$ which is a coordinate plane in the new system of axes. Hence the original equations must also represent this plane, and we get that any two equations of first degree in four variables represents a plane.

We shall now proceed with the problem of finding out this plane geometrically.

The general equations of a plane in four dimensions may be written,

$$\left. \begin{aligned} l\alpha + m\beta + n\gamma + p\delta &= a \\ l'\alpha + m'\beta + n'\gamma + p'\delta &= b. \end{aligned} \right\} \dots \dots (A)$$

By eliminating β and δ in succession between the two equations we can reduce them to the form,

$$\begin{aligned} \beta &= a\alpha + c\gamma + e & (i) \\ \delta &= b\alpha + d\gamma + f & (ii) \end{aligned} \dots (X).$$

A special advantage of writing the equations in this form is that a (1—1) correspondence is established between the possible planes in a four-dimensional domain and the equations obtained by varying the constants. Such advantage does not belong to the equations (A) where all the variables are present in both. The same plane may, in that case, be represented by different pairs of equations.

Since with every change of the constants of the equations (X) a new plane is arrived at, and there are six of these constants, the number of planes possible in four dimensional space is six-fold infinity.

Turning now to the equations we see that (i) represents a plane in three-dimensional geometry (this is really the intersection of the two solids of the first degree $\beta = a\alpha + c\gamma + e$ and $\delta = 0$). Thus the solid (i) passes through the plane $\beta = a\alpha + c\gamma + e$ in (α, β, γ) space. Again (ii) is a solid whose section by $\delta = 0$ is the plane in three dimensions $b\alpha + d\gamma + f = 0$. Now in (α, β, γ) space $\beta = a\alpha + c\gamma + e$ and $b\alpha + d\gamma + f = 0$ together represent a straight line, which being common to the two planes is common to the two solids (i) and (ii). Hence the plane (X) has this line lying on it.

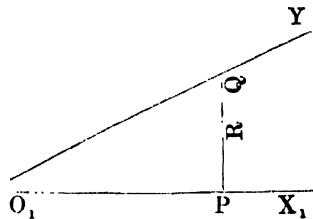
So much for the (α, β, γ) space. In (α, γ, δ) space, similarly, we get the planes of section to be $a\alpha + c\gamma + e = 0$ and $\delta = b\alpha + d\gamma + f$ and these determine by their intersection a straight line which is common to (i) and (ii) and, therefore, to the plane (X).

Now these straight lines have the point given by $a\alpha + c\gamma + e = 0$ and $b\alpha + d\gamma + f = 0$ in the plane of reals, common. Hence these are coplanar and as the surface of intersection of two solids of first degree has been shown to be a plane, the equations (X) are the analytical equivalent of the plane defined by these two straight lines.

We might, however, show that every point in the plane determined by these two straight lines lies on both the solids and, therefore, on their surface of intersection, and thus get an alternative proof of the fact that the intersection of two solids of first degree is a plane.

To show that the plane determined by the line $b\alpha + d\gamma + f = 0$ and $\beta = a\alpha + c\gamma + e$ in the (α, β, γ) space, and the line $a\alpha + c\gamma + e = 0$ and $\delta = b\alpha + d\gamma + f$ in the (α, γ, δ) space, is the surface of intersection of (i) & (ii).

Let O_1X_1 and O_1Y_1 be the straight lines. Then if $(\alpha', \beta', \gamma', 0)$ be any point P on O_1X_1 and $(\alpha'', 0, \gamma'', \delta'')$ any point Q on O_1Y_1 , any point, R on PQ will be given by $(\alpha' + k\alpha'', \beta', \gamma' + k\gamma'', k\delta'')$.



But if P & Q lie on both the solids, the point R will also lie on them, and hence on their surface of intersection. But by varying the points on the lines O_1X_1 and O_1Y_1 and k , we can make R coincide with any point in the plane; hence the plane $X_1O_1Y_1$ lies altogether on the surface of intersection of (i) & (ii); or the two coincide.

§3. The planes of examination

We now pass on to notice some of the most important particular cases of planes in four dimensions.

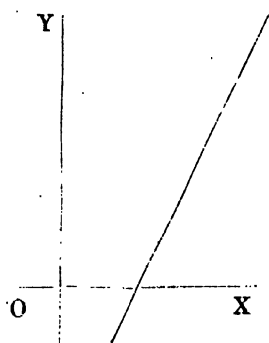
Where one of the two equations defining the plane contains two of the variables (α and γ), and the other, the other two (β and δ) we get what may be termed a *plane of examination*.

The equations might be written

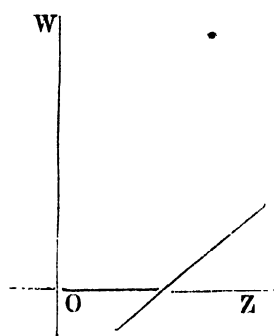
$$\left. \begin{aligned} \gamma &= m\alpha + c \dots (iii) \\ \delta &= m'\beta + d \dots (iv) \end{aligned} \right\} \dots \dots (Y)$$

We now proceed to a geometrical consideration of these equations in defining the plane formed by them.

Let us take two planes, the real and the imaginary, and draw in them the straight lines



Plane of reals



Plane of imaginaries

which have the given equations (iii) & (iv) and which are really the sections of the solids (iii) and (iv) by these planes.

Definitions

The real and the imaginary plane together may be called the *basic planes*.

The complex point $(a + i\beta, \gamma + i\delta)$ determined by the points (a, γ) and (β, δ) in the real and imaginary planes is said to be *formed by their association*. The points in the basic planes may be termed its *components*.

The plane formed by the association of a line in the real plane and a line in the imaginary plane is that defined by lines drawn parallel to them through the complex point formed by the association of a point on the real line of association and a point on the imaginary line of association.

These lines of association are, of course, at right angles.

Analytically, the equations to this plane are given by (iii) and (iv) together, for these two together represent a plane. And the (a, γ) coordinates of the plane satisfy (iii) and the (β, δ) coordinates satisfy (iv). Hence both (iii) and (iv) pass through this plane, which therefore must coincide with their plane of intersection.

Hence we obtain, that there is only one plane formed by the association of a line in the real plane and a line in the imaginary plane.

We now prove some important propositions with regard to these planes of examination. For convenience the word "plane" in what follows will mean a "plane of examination."

Proposition I

Two planes will intersect in the complex point formed by the association of the points of intersection, in the basic planes, of the lines of association of the planes.

This follows from the definition of the planes. If B & D be the points of intersection of the lines of association, the complex point (B, D) formed by the association of B & D, lies on both the planes and is their point of intersection.

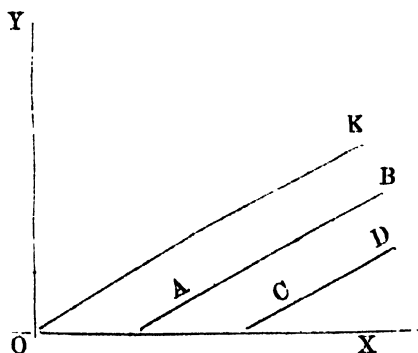
Hence it follows that two planes will, in general, intersect in only a single point; for the lines in the basic planes intersect in only a single point unless coinciding.

Proposition II

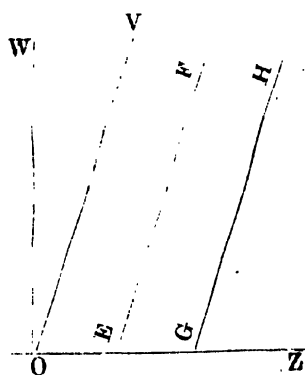
Two planes are parallel when they are formed by the association of lines which are parallel straight lines in their planes of reference.

Definition of parallelism

Two planes are parallel when the line at infinity of one coincides with the line at infinity of the other



Plane of reals.

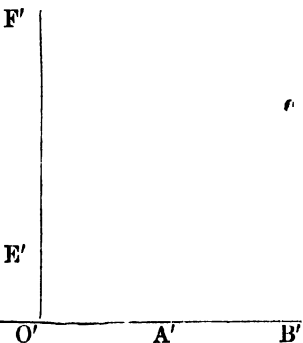


Plane of imaginaries.

Let (AB, EF) and CD, GH) be the two planes and AB be parallel to CD and EF to GH. By Prop. I the point of intersection is the

complex point formed by the association of the points at infinity along AB & EF .

If we draw the plane of examination F' (AB, EF) having as origin a point O' formed by the association of a point on AB and a point on EF , the complex point of intersection will be that determined by a point at infinity along $A'B'$ and a point at infinity along $E'F'$ (these being lines parallel to AB and EF through O'). But these coordinates do not define a single unique point but



an infinity of points infinitely distant and lying on the line at infinity. Hence the points of intersection lie on the line at infinity on (AB, EF) . Similarly they lie on the line at infinity on the plane (CD, GH) . Thus the two planes have identical lines at infinity *i.e.*, are parallel.

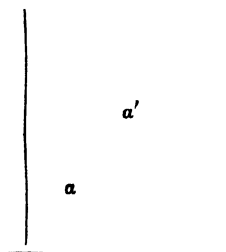
Proposition III

Two planes are also parallel when one pair of parallel lines of association become coincident.

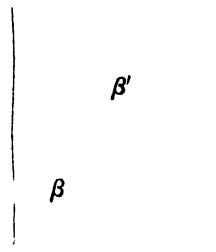
This follows from the preceding proposition when we remember that the line at infinity includes points, one of whose coordinates is finite. [These, of course, are one or other of the two points at infinity lying on the axes].

Proposition IV

Of the two systems of doubly infinite planes which pass through two different points, among corresponding parallel planes there is only a single pair which is coincident.



Plane of reals



Plane of imaginaries.

Let (α, β) and (α', β') be the points through which the systems of planes are drawn. The corresponding parallel planes are those which have their axes parallel (*i.e.*, are formed by the association of lines which

are parallel in their basic planes). In the case in which the parallel axes get coincident (*i.e.*, in the case of the plane $(\alpha\alpha', \beta\beta')$ we have coincident planes. The plane of examination $(\alpha\alpha', \beta\beta')$ is common to both the systems.

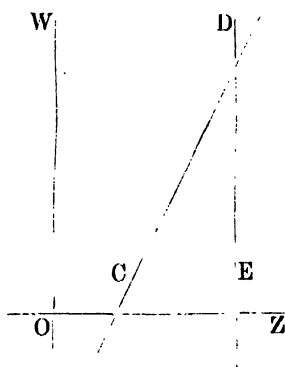
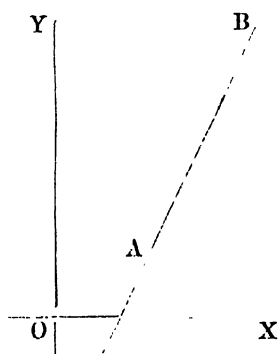
Proposition V

Two planes will meet in a single point at infinity when one of the lines of association of one is parallel to that of the other.

Prop. I gives us that the point of intersection is that formed by the association of the finite points of intersection in one basic plane, and a point at infinity in the other. When we draw one of the planes we find that the point of intersection is at infinity along an axis.

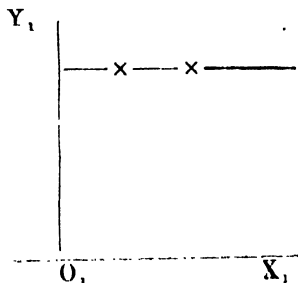
Proposition VI

If a line of association of one plane coincides with that of the other, the two planes intersect in a line parallel to the coincident line of association.



Let the coincident line of association be in the real plane and let it be AB, and let CD and DE be the imaginary lines of association intersecting at D. Then Prop. I gives the points of intersection as those formed by association of every point of AB with D.

Let us draw the plane (AB, CD) , and let it have as origin O_1 found in the usual way. Then in this plane the points of intersection will have the same y -coordinate, and, therefore the line of intersection will be parallel to O_1X_1 and hence to the coincident line of association AB.



The analytical proof is also interesting. Let the planes be

$$\left. \begin{aligned} \gamma &= ma + c, \\ \delta &= m_1\beta + d_1, \end{aligned} \right\} \quad \left. \begin{aligned} \gamma &= ma + c, \\ \delta &= m_2\beta + d_2. \end{aligned} \right\}$$

The equations to the common line of intersection are,

$$\left. \begin{aligned} \gamma &= ma + c, \\ \delta &= m_1\beta + d_1, \\ \delta &= m_2\beta + d_2. \end{aligned} \right\}$$

And the straight line is obviously one parallel to the line $\gamma = ma + c$ in the real plane, through the point $(0, \beta, 0, \delta)$ where β and δ are determined from the last two equations.

§4. *The plane of the first degree*

We now come to another particular case of the general equations to a plane. It is furnished by the general equation of the first degree in two variables.

The most general form of the equation of the first degree is $y = mx + c$.

Splitting up the real and imaginary parts after substituting $a + i\beta$ for x and $\gamma + i\delta$ for y ,

$$\gamma = ma + c \dots (a) \qquad \delta = m\beta \dots (b).$$

This shows that the equation whose manifestation is a straight line in the plane of reals is, in reality, a plane. We see also that it belongs to the class of the planes of examination. Such a plane is termed a *plane of the first degree*.

Planes of first degree are, however, particular cases of the planes of examination.

For, from the equations (a) & (b) we see (a) that the lines of association are inclined at the same angles to the coordinate axes in their respective basic planes, and (b) that the line of association in the imaginary plane passes through the origin. These conditions doubly limit the possible number of planes and we get the totality of such planes to be only a two-fold infinity. The single equation in two variables to the surface also shows this to be the case.

The two following propositions with regard to planes of first degree are of importance.

Proposition A

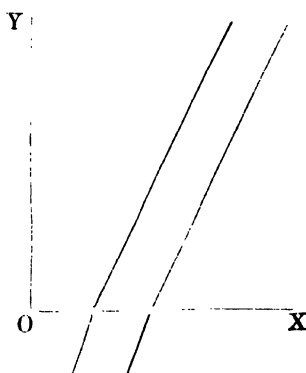
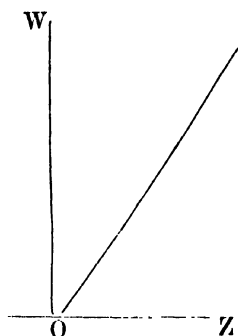
Equations of first degree in two variables which represent parallel lines in the plane of reals are really parallel planes meeting in a single infinity of points, situated altogether at infinity.

Let us take the two equations

$$y = m'x + c \dots (c)$$

and

$$y = m'x + c' \dots (d)$$



For equation (c) the lines of association are,

$$\left. \begin{aligned} \gamma &= m\alpha + c, \\ \delta &= m\beta. \end{aligned} \right\} \dots$$

For equation (d) they are,

$$\left. \begin{aligned} \gamma &= m\alpha + c', \\ \delta &= m\beta \end{aligned} \right\} \dots$$

Since the planes have parallel lines of association in the real plane, and coincident lines of association in the imaginary plane they represent parallel planes by Prop. III p. 180.

The second part of the proposition is proved from the definition of parallel planes.

A complementary proposition with regard to all other equations of first degree is given by,

Proposition B

In all other cases they represent planes having only a single point of intersection, which is in the real plane.

This follows from the fact that two first degree equations in two variables can only have a real solution. The proposition might also be proved by the theory of planes of examination by the help Prop. I p. 179. The lines in the imaginary plane pass through the origin; the lines in the real plane intersect at a definite point. Hence the point of intersection is obtained by associating the definite point in the plane of reals with the origin in the plane of imaginaries. Hence the point lies in the real plane, and is the intersection of the real lines of association.

The plane of the first degree with complex coefficients

We shall now consider the first degree equation in two variables, where the constants are complex quantities, and see what the equation represents under these circumstances.

Let the equation be

$$y = (A + iB)x + C + iD,$$

or

$$\gamma + i\delta = (A + iB)(\alpha + i\beta) + C + iD.$$

Splitting up real and imaginary parts,

$$\gamma = A\alpha - B\beta + C,$$

$$\delta = B\alpha + A\beta + D.$$

Hence the plane belongs to the most general class though it is a special case and contains only four constants.

The section of this plane by the solid $\delta=0$, is the straight line,

$$\left. \begin{aligned} \gamma &= A\alpha - B\beta + C, \\ 0 &= B\alpha + A\beta + D, \end{aligned} \right\} \dots$$

in the (α, β, γ) space.

The direction-cosines of the line are proportional to

$$A, -B, A^2 + B^2.$$

Since every line in the (α, β, γ) space is perpendicular to the δ -axis, the fourth direction cosine of the line is 0.

Hence the direction-cosines are proportional to,

$$A, -B, A^2 + B^2, 0.$$

Similarly the section of the plane by $\gamma=0$ is the line

$$\left. \begin{aligned} O &= A\alpha - B\beta + C, \\ \delta &= B\alpha + A\beta + D, \end{aligned} \right\} \dots$$

in the (α, β, δ) space.

The direction-cosines are similarly proportional to,

$$B, A, 0, A^2 + B^2.$$

Hence applying $ll' + mm' + nn' + pp' = 0$ we see that the two characteristic lines are at right angles to each other.

Hence, the same method as in the case of the most general equation gives the geometrical location of the plane. In this case the characteristic lines determining the plane are found to be at right angles. This will be of use in determining curve-sections of surfaces and solids in such a plane.

§5. Applications.

The ground having been thus prepared, we shall now deal with the problem of the intersections of equations in two variables, where the solutions are not available in the real plane.

Problem I.

To find the points or lines of intersection of the surfaces given by $x^2 + y^2 = 0$ and $x^2 + y^2 - a^2 = 0$

Put

$$x = a + i\beta, y = \gamma + i\delta, \text{ and the equations become,}$$

$$(a + i\beta)^2 + (\gamma + i\delta)^2 = 0. \quad (a + i\beta)^2 + (\gamma + i\delta)^2 = a^2.$$

whence we get

$$\left. \begin{aligned} a^2 - \beta^2 + \gamma^2 - \delta^2 &= 0 \dots (1) \\ a\beta + \gamma\delta &= 0 \dots (2). \end{aligned} \right\} \quad \left. \begin{aligned} a^2 - \beta^2 + \gamma^2 - \delta^2 &= a^2 \dots (3). \\ a\beta + \gamma\delta &= 0 \dots (4). \end{aligned} \right\}$$

Equations (2) and (4) are identical, and we have a case of curve-intersection of the two surfaces,

(1) & (3) are inconsistent together unless $\alpha^2 + \gamma^2$ and $\beta^2 + \delta^2$ tend to become infinite, approaching each other in a ratio of equality.

From (2) we have,

$$\frac{\alpha}{\gamma} = \frac{-\delta}{\beta} = k \text{ suppose... (5).}$$

This shows that $\alpha, \beta, \gamma, \delta$ must all be infinite for the points of intersection. From (5) we find that the points at infinity where the curves intersect are of the form $(\alpha + i\beta, i\alpha - \beta)$ or (X, iX) where X gets infinite along a particular radius vector given by the ratio of α and β . Since, however, this ratio is indefinite, we have the case of a single infinity of points at infinity. Hence the surfaces intersect in a *curve* at infinity. We shall now determine whether these *touch* at infinity all along the curve. The sections in the real-imaginary and imaginary-real plane seem to suggest that this might be the case.

Before doing so, however, let us develop a method of obtaining an infinity of planes where we may get curve-sections of the surfaces. Let us analyse the equations (1) & (2) viz,

$$\alpha^2 - \beta^2 + \gamma^2 - \delta^2 = 0 \dots (1)$$

$$\alpha\beta + \gamma\delta = 0 \dots (2)$$

(2) by itself denotes a surface, and the planes obtained by giving to k all real values in,

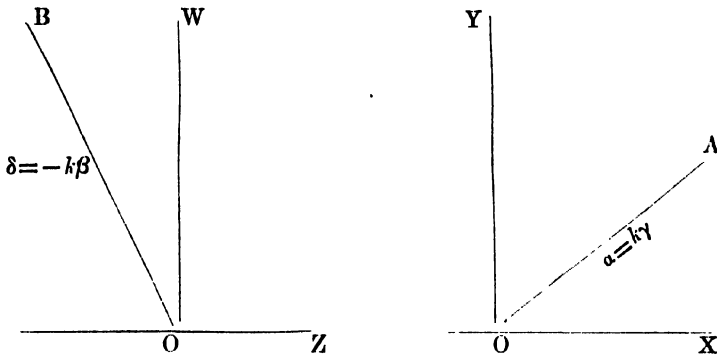
$$\frac{\alpha}{\gamma} = \frac{-\delta}{\beta} = k \dots (5).$$

lie entirely on the surface. Hence the surface consists entirely of these planes, (which, it will be seen, are planes of examination). This is a case analogous to the generating lines of a ruled surface. They may be termed the *generating planes* of the solid.

Thus the intersection of these planes with equation (1) will give the entire surface. The curves in these, in their totality, represent the whole surface.

Now for a definite value of k , the equations (5) represent two equations. These two equations combined with (1) give us three equations among the four variables. We have thus for their intersection a certain curve, and we proceed to analyse this curve and see what it geometrically represents,

In order to determine the section in the plane $a=k\gamma$ $\delta=-k\beta$, for any value of k the following artifice may be adopted.



Plane of imaginaries.

Plane of reals.

The plane is formed by the association of the lines OA & OB in the real and imaginary planes whose equations are $a=k\gamma$, $\delta=-k\beta$.

Take for axes the lines OA & OB which are at right angles to each other. A unit length along OA will have the coordinates,

$$a = \frac{k}{\sqrt{1+k^2}}, \gamma = \frac{1}{\sqrt{1+k^2}}$$

and a unit length along OB,

$$\beta = \frac{1}{\sqrt{1+k^2}}, \delta = \frac{-k}{\sqrt{1+k^2}}.$$

In order to determine the curve in the plane our method will be to take any point (m, m') referred to these lines as axes, and then to express the original coordinates in terms of m & m' . The substitution of these in the given equation will give us a relation between m & m' which will be the curve required in the plane.

The point (m, m') in this plane has for its coordinates,

$$a = \frac{mk}{\sqrt{1+k^2}}, \gamma = \frac{m}{\sqrt{1+k^2}}$$

$$\beta = \frac{m'}{\sqrt{1+k^2}}, \delta = \frac{-m'k}{\sqrt{1+k^2}}.$$

and substituting these in equation (1) viz.

$$a^2 - \beta^2 + \gamma^2 - \delta^2 = 0.$$

we get

$$\frac{m^2 k^2}{1+k^2} - \frac{m'^2}{1+k^2} + \frac{m^2}{1+k^2} - \frac{m'^2 k^2}{1+k^2} = 0.$$

or

$$x^2 - m'^2 = 0.$$

[We would have got the same result if we had substituted these values in $x^2 + y^2 = 0$, where $x = a + i\beta$, $y = \gamma + i\delta$].

The equation obtained is independent of k and we see that in all these planes the section is the same; viz the pair of straight lines $m^2 - m'^2 = 0$.

In the case of the equation $x^2 + y^2 = a^2$, we see in an exactly similar way that the section is the rectangular hyperbola, $m^2 - m'^2 = a^2$.

Since the planes of examination in both these cases are identical, the two surfaces touch each other at infinity at two points in each of these planes, where the m & m' coordinates are in the ratio of equality, and, therefore, the x & y coordinates are of the form (X, iX) , or (iX, X) . This happens in all the planes obtained by varying k , and as these planes, in their totality, contain the entire surface defined by the two equations, these touch each other all along the single infinity of points thus obtained; i.e., they touch each other all along their curve of intersection.

In the case of the equation $x^2 + y^2 = -a^2$ the same process will give the section in the identical planes of examination to be $m^2 - m'^2 + a^2 = 0$ and this represents the conjugate rectangular hyperbola, and the surface touches in the curve at infinity the other two surfaces.

The same is true of all the equations of the form $x^2 + y^2 = a^2$ obtained by varying a^2 , which are concentric circles in the plane of reals and concentric rectangular hyperbolas in the single infinity of planes of examination $a = k\gamma$, $\delta = -k\beta$. And we deduce that they all touch in their common curve at infinity at every point of which the x & y coordinates are in the ratio of 1: i .

Problem II.

To determine the points and lines of intersection of the surfaces given by $x^2 + y^2 = a^2$, and $(x-h)^2 + y^2 = b^2$.

The equations become, when broken up into real and imaginary parts,

$$a^2 + \gamma^2 - \beta^2 - \delta^2 = a^2 - (1') \quad (a-h)^2 + \gamma^2 - \beta^2 - \delta^2 = b^2 - (3')$$

$$a\beta + \gamma\delta = 0 - (2') \quad (a-h)\beta + \gamma\delta = 0 - (4')$$

The finite points of intersection might be obtained by solving the equations. For the points of intersection at infinity we shall adopt the analysis of the preceding example.

As before, the sections in the planes, $\frac{a}{\gamma} = \frac{-\delta}{\beta} = k$, in the case of the equations (1') & (2') and those in the planes $\frac{a-h}{\gamma} = \frac{-\delta}{\beta} = k$ in the case of the equations (3') & (4'), are rectangular hyperboles whose asymptotes are $m^2 - m'^2 = 0$.

Now the planes $a = k\gamma$, $\delta = -k\beta$ and $(a-h) = k\gamma$, $\delta = -k\beta$ are parallel (by Prop. III p. 180). They intersect in the line at infinity in their planes.

[That the planes intersect in a line is also apparent from the fact that their equations are equivalent to the following three equations:— $a = k\gamma$, $a-h = k\gamma$ and $\delta = -k\beta$.]

The parallel asymptotes in the parallel planes, and therefore also the rectangular hyperbolas intersect in points at infinity which lie on this line at infinity. This happens in the case of the single infinity of planes obtained by giving k all real values. Thus we obtain that the surfaces intersect in two finite points and a single infinity of points at infinity, or a curve at infinity. The coordinates of the points at infinity along the system of planes $a = k\gamma$, $\delta = -k\beta$ are $\{A + iB, i(A + iB)\}$ where A & B have infinite values in any ratio: *i. e.* $A + iB$ may become infinite along any vector.

[We also see why there should be a curve of intersection of the surfaces, from a consideration of the equations. For the four equations are really equivalent to the three:— $a = \infty$, $a^2 - \beta^2 + \gamma^2 - \delta^2 = 0$ and $a\beta + \gamma\delta = 0$, a curve altogether at infinity].

The problem is very similar in the case of the equations

$$\alpha^2 + \gamma^2 - \beta^2 - \delta^2 = a^2 - (1'') \quad (\alpha - h)^2 + (\gamma - k)^2 - \beta^2 - \delta^2 = b^2 - (3'')$$

$$\alpha\beta + \gamma\delta = 0 - (2'') \quad (\alpha - h)\beta + (\gamma - k)\delta = 0 - (4'')^3$$

In the case of equations (1'') and (2'') the single infinity of planes $\frac{\alpha}{\gamma} = \frac{-\delta}{\beta} = k$ have curve-sections of the surface which are rectangular hyperbolas. The corresponding planes are parallel by Prop III p. 180 and the asymptotes being parallel lines in these planes intersect in two points at infinity on the line at infinity on both. The curve-sections, consequently, intersect and the surfaces have points of intersection in each of the single infinity of planes obtained by giving k all real values. Regarding the finite points of intersection the ordinary methods suffice. Combining all these we get.

(i) The equations which in the plane of reals are concentric circles, are surfaces which touch at all points on a curve at infinity whose x, y coordinates are in the ratio $1 : \pm i$. These points have coordinates of the form $[A + iB, \pm i(A + iB)]$ where A & B are infinite and different points are obtained by varying the ratio in which they become infinite.

(ii) All equations representing circles in the plane of reals have a common curve of intersection which is the circle at infinity. Any two of these have, besides two finite points of intersection.

And now we can see why it is that *two circles can never intersect in more than two points*, whereas two conics will generally have four points of intersection. The corresponding algebraical equations with which the circles have been associated have two finite and two infinite solutions, and the points corresponding to the infinite solutions are always outside the plane of reals. The finite solutions give rise to finite points and where these are real the circles intersect. But the infinite solutions give rise to the curve at infinity whose x & y coordinates are in the ratio of $1 : \pm i$, and which is, therefore, absolutely outside the plane of reals.

Two concentric circles can never intersect.—For the corresponding algebraical equations with which they are associated have two pairs

of coincident infinite solutions, giving rise to the circle at infinity twice. Hence these can have no finite points of intersection. The infinite points are outside the plane of reals.

The following examples illustrate the method by which we can bring to view the surface represented by the general equation of the second degree in two variables, the consideration of which will appear in the next Chapter.

Problem III.

To determine a single infinity of planes which intersect the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in curves, and to find the equations to the curve-sections in these planes.

The first portion is solved by a method similar to the preceding. On separation of real and imaginary parts, the equation splits up into,

$$\left. \begin{aligned} \frac{\alpha^2 - \beta^2}{a^2} + \frac{\gamma^2 - \delta^2}{b^2} &= 1 \\ \frac{\alpha\beta}{a^2} + \frac{\gamma\delta}{b^2} &= 0 \end{aligned} \right\} - (A).$$

From the second equation we have our system of planes to be,

$$\frac{\gamma}{a} = -\frac{b^2}{a^2} \cdot \frac{\beta}{\delta} = m.$$

or

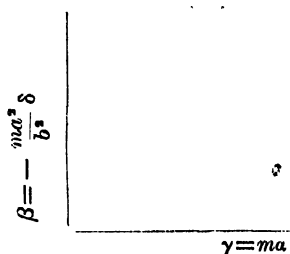
$$\gamma = m\alpha, \quad \beta = -\frac{ma^2}{b^2} \delta.$$

To find the equation in this plane we take the lines of association $\gamma = m\alpha$ in the real plane, and $\beta = -\frac{ma^2}{b^2} \delta$ in the imaginary plane as our axes.

If (k, k') be any point in the plane with reference to these axes, the $(\alpha, \beta, \gamma, \delta)$ coordinates of the point are,

$$\alpha = \frac{k}{\sqrt{1+m^2}}, \gamma = \frac{mk}{\sqrt{1+m^2}}$$

$$\beta = -\frac{ma^2k'}{\sqrt{b^4+m^2a^4}}, \delta = \frac{b^2k'}{\sqrt{b^4+m^2a^4}}.$$



On substitution of these values in the first of equations (A) we get the locus to be,

$$\frac{k^2}{a^2b^2(1+m^2)} - \frac{k'^2}{b^4+m^2a^4} = \frac{1}{m^2a^2+b^2}$$

We thus see that the sections are different in different planes. The aggregate of all these curves is the surface itself. Its manifestation in the plane of reals is what we ordinarily associate with the representation of the equation. We see, however, that all the curves are conic sections.

We take another problem to illustrate the case where the lines of association do not pass through the origin.

Problem IV.

To solve a similar problem in the case of the equation $y^2 = 4ax$.

The equation may be written,

$$(\gamma + i\delta)^2 = 4a(\alpha + i\beta)$$

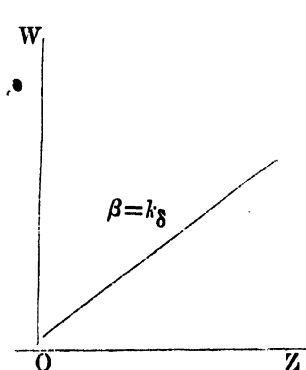
which splits up into,

$$\left. \begin{aligned} \gamma^2 - \delta^2 &= 4a\alpha \\ \gamma\delta &= 2a\beta \end{aligned} \right\} \dots (A').$$

From the latter we get the equations of the planes of examination as,

$$\frac{\gamma}{2a} = \frac{\beta}{\delta} = k$$

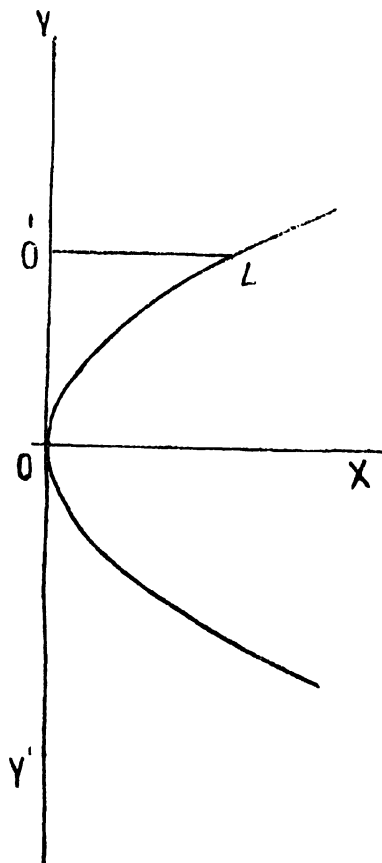
and the lines of association in the real and imaginary plane are,
 $\gamma = 2ak$, $\beta = k\delta$.



Plane of imaginaries.

Let O_1 be the point where the real line of association intersects OY ; and (OO_1) (i.e., & formed by association of O & O_1), be the origin of coordinates in our plane of examination.

The point (m, m') will have its coordinates,



Plane of reals,

$$a = m, \gamma = 2ak$$

$$\beta = \frac{km'}{\sqrt{1+k^2}}, \delta = \frac{m'}{\sqrt{1+k^2}}.$$

Substituting in the first of equations (A'), the equation to the complementary curve becomes,

$$4a^2k^2 - \frac{m'^2}{1+k^2} = 4am$$

or

$$\frac{m'^2}{1+k^2} = -4a(m-ak^2)$$

Transferring the origin to the point $(ak^2, 0)$ with reference to new axes, or to

$$a=ak^2, \beta=0, \gamma=2ak, \delta=0,$$

which is the point where $O'L$ intersects the parabola in the real plane, the equation to the complementary parabola becomes,

$$\frac{m'^2}{1+k^2} = -4am.$$

This parabola meets the parabola in the real plane, and has its axis in the opposite direction. All points along $O'L$ where m is positive are within the principal curve, and for negative values of m the points are within the curve in the plane of examination.

PHYSICS

VII. Investigations on the Acoustics of the Pianoforte : I

By

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(Received for publication on 1st May, 1923.)

1. Introduction.

In a paper published in the Proceedings of the Indian Association for the Cultivation of Science, Vol. VII, parts I and II, Mr. Panchanan Das has given an equation for the pressure exerted by the pianoforte hammer for the case when the hammer is rigid. He obtained the solution by the functional method of Kaufmann¹ considering the discontinuous changes in the pressure of the hammer due to the successive reflections of the impulse from the ends of the string. Prof. C. V. Raman and B. Banerji² obtained the solution of the same problem in a different manner by regarding the motion during impact as that of a loaded string. Mr. Das proceeded on a line similar to that given by Prof. Love³ for the longitudinal vibrations of a rod struck at the end.

The equation obtained by Mr. Das for the pressure exerted by the hammer is

$$F = 2\rho v_0 c \left[e^{-kct} + e^{-k(ct-2a)} \left\{ 1 - k(ct-2a) \right\} + e^{-k(ct-4a)} \left\{ 1 - 2k(ct-4a) + \frac{k^2}{2!} (ct-4a)^2 \right\} \dots \right] \dots \quad (1)$$

¹ Ann. der Physik, Vol. 54 (1895).

² Proc. Roy. Soc. A., Vol. 97 (1920).

³ Love's "Theory of Elasticity," p. 411, Art. 281 (2nd Edn.)

where

ρ = the linear density of the string

m = the mass of the hammer

$k = 2\rho/m$

c = velocity of propagation of transverse wave on the string

v_0 = the initial velocity of the hammer

a = the distance of the striking point from the nearer end

t = variable time.

He considered the total duration of contact of the hammer as divided up into a series of epochs determined by $0 < ct < 2a$, $2a < ct < 4a$, $4a < ct < 6a$ and so on, so that each of these epochs corresponds to the time taken by a pulse starting from the striking point in returning to the striking point after reflection from the nearer end.¹ Thus the pressures in the successive epochs are:—

$$\left. \begin{aligned} F_1 &= 2\rho v_0 c e^{-kct} \\ F_2 &= 2\rho v_0 c \left[e^{-kct} + e^{-k(ct-2a)} \right] \left\{ 1 - k(ct-2a) \right\} \\ F_3 &= 2\rho v_0 c \left[e^{-kct} + e^{-k(ct-2a)} \right] \left\{ 1 - k(ct-2a) \right\} \\ &\quad + e^{-k(ct-4a)} \left\{ 1 - 2k(ct-4a) + \frac{k^2}{2!} (ct-4a)^2 \right\} \end{aligned} \right\} \quad (2)$$

It will thus be seen, that at the end of every epoch, the pressure increases by a sudden jump of magnitude $2\rho v_0 c$. Now we can determine the amplitudes of the different harmonics if we know the nature of the force acting on the string. For if F be the force acting on the string at a point distant a from one end, then using the notation adopted in Art. 130 of Lord Rayleigh's 'Theory of Sound,' Vol. 1, we find

¹ In his work Mr. Das considered reflection from the nearer end only, as the hammer leaves the string before the reflected pulse from the farther end reaches it—a supposition which is valid only for striking points near the ends.

that the amplitudes of the different harmonics are given by

$$\phi_n = \frac{2}{\pi c \rho} \sin \frac{\pi a}{l} \int_0^t F \sin n(t-t') dt \quad \dots (3)$$

where the integration is to be made over the whole duration of contact. Thus we can proceed to calculate the amplitudes of the different harmonics by substituting for F the value given by the equation. In the present paper it will be shown that the amplitudes of the different harmonics as calculated from the equation agree with those found experimentally.

2. *The calculation of the amplitudes of the different harmonics.*

In order to calculate the amplitudes of the different harmonics we have to integrate the pressure equation over the whole of the duration of contact of the hammer with the string. But as the actual integration of the equation involves considerable mathematical labour, a mechanical and graphical method was employed in the integration of equation (3) given above, which considerably simplified the calculation. The method followed is that given below.

The pressures at successive epochs given by equation (2) are plotted against time, thus giving the variation of pressure of the hammer with time throughout the whole of the duration of contact of the hammer—(the point where the curve cuts the axis of time gives the duration of contact). Now since the magnitude of the discontinuous change in the pressure exerted by the hammer is the same at the end of each epoch, the plotting of the pressure-time curve is much simplified by calculating the initial discontinuity and the pressures at the end of every epoch at which the pressure suffers a sudden discontinuous change, and joining the intermediate portions for which the pressure decreases continuously by a continuous

straight line. The error committed in this way is very small as the actual curve differs very little from a straight line. Thus, at the time $t=0$, the pressure suddenly increases to $2\rho v_0 c$, and then continuously decreases until the reflected pulse from the end arrives at the hammer which takes place at the time $ct=2a$, when the pressure again increases suddenly by the same amount, *viz.*, $2\rho v_0 c$. Just at the end of the first epoch, *i.e.*, at the time $ct=2a$, the pressure is $F_1=2\rho v_0 c e^{-kct}$ and then suddenly increases by an amount equal to $2\rho v_0 c$, and again falls off continuously until the second reflected pulse reaches it, which takes place at the end of the time $ct=4a$, when it again increases discontinuously by the same amount, and again falls off continuously until the third reflected pulse reaches it, and so on, till the impact ceases. Having obtained the pressure time curve between the time $t=0$ to the time when the curve cuts the axis of time, we proceed to calculate the value of the expression.

$$\phi_s = \frac{2}{\pi c \rho} \sin \frac{\pi a}{l} \int_0^t F \sin n(t-t') dt'$$

Now the expression inside the integral sign can be broken up into two expressions

$$\int_0^t F \sin n(t-t') dt' = \int_0^t F \sin nt \cos nt' dt' - \int_0^t F \cos nt \sin nt' dt'$$

and taking $\sin nt$ and $\cos nt$ outside the integral sign, the expression for ϕ_s becomes

$$\phi_s = \frac{2}{\pi c \rho} \sin \frac{\pi a}{l} \left[\sin nt \int_0^t F \cos nt' dt' - \cos nt \int_0^t F \sin nt' dt' \right] \dots \quad (1)$$

Thus to calculate the values of ϕ , we multiply each ordinate of the pressure-time graph by $\cos nt'$ and $\sin nt'$ respectively and obtain two new curves with these values as ordinates. The areas of these curves between the origin and the point where they cut the axis of time give the values of the two integrals in equation (4). The areas of these curves were measured by means of a planimeter and the amplitudes of the different harmonics were obtained by putting $s = 1, 2, 3$, etc., respectively. In the numerical computation, those values of p, v_0, c and k were used which were afterwards employed in the experimental determination described below. The calculated values of the amplitudes up to the tenth harmonic are given in Tables I and II for two different striking points, viz., $l/10$ and $l/9$.

3. *Experimental determination of the amplitudes of the different harmonics.*

An experimental verification of the above results was attempted in the following way. A steel wire 150 cm. long was stretched over the bridges fixed on the table. The linear density of the wire was .081 gm. per centimetre. A mechanical hammering arrangement was made by mounting a small solid brass cylinder at the end of a light pivoted shaft, which could be caused to impinge transversely on the wire. The effective mass of the hammer used was 20.5 grammes. The amplitudes of the different harmonics were measured by means of an observing microscope. For measuring the amplitude of the fundamental, the microscope was focussed on the middle of the string and the hammer was caused to strike at the points for which the calculations were made, viz., $l/10$ and $l/9$ respectively. To measure the amplitude of the octave, the microscope was focussed at $l/4$, and the string was damped at the middle immediately after the impact so as to stop the fundamental. Similarly, for other harmonics the string was damped at the corresponding node immediately after impact,

and observed at an antinode. The different amplitudes as measured by the microscope were then corrected for the amplitudes of other harmonics for which the point of observation is also an antinode. Thus from the measured amplitude of the fundamental were subtracted those of the 3rd, 5th and so on; similarly for others. In the following tables are given the observed values of the amplitudes of the different harmonics along with those calculated from the equation for two different striking points, *viz.*, $l/10$ and $l/9$. The measurements were made up to the tenth harmonic for both the striking points.

TABLE I.

Striking point at $l/10$.

Harmonics.				Observed amplitudes.	Calculated amplitudes.
Fundamental	·3700 cm.	·3975 cm.
Octave	·0588 ..	·0593 ..
Third	·0433 ..	·0433 ..
Fourth	·0130 ..	·0131 ..
Fifth	·0078 ..	·0079 ..
Sixth	·0100 ..	·0147 ..
Seventh	·0020 ..	·0022 ..
Eighth	·0157 ..	·0173 ..
Ninth	·00112 ..	·00128 ..
Tenth	·0 ..	·0 ..

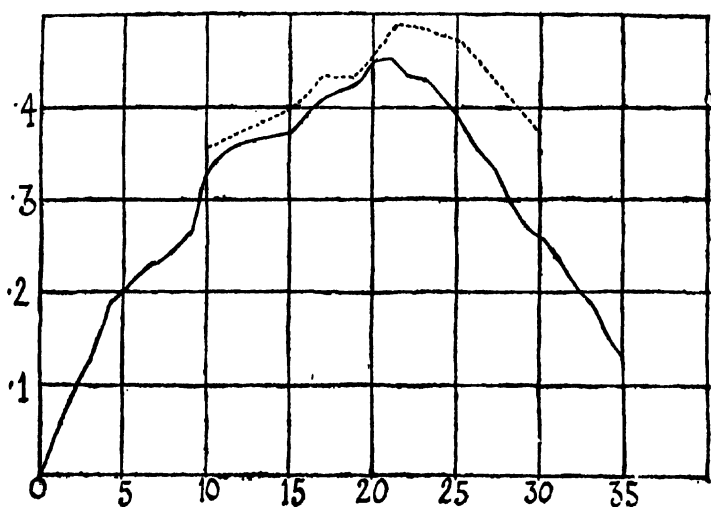
TABLE II.
Striking point $l/9$.

Harmonics.				Observed amplitudes.	Calculated amplitudes.
Fundamental	·4240 cm.	·4322 cm.
Octave	·0345 „	·0365 „
Third	·0167 „	·0173 „
Fourth	·0217 „	·0240 „
Fifth	·0046 „	·0043 „
Sixth	·0029 „	·0027 „
Seventh	·0071 „	·0068 „
Eighth	·0020 „	·0023 „
Ninth	·0 „	·0 „
Tenth	·0022 „	·0025 „

4. Variation of the amplitudes of the different harmonics with the striking point.

In order to test the equation more fully, the amplitudes of the fundamental were determined for different striking points along the string both from the equation as well as experimentally. Theoretical calculations were made for seven different striking points along the string, *viz.*, $l/15$, $l/10$, $l/9$, $l/8$, $l/7$, $l/6$ and $l/5$. Beyond the point $l/5$ the equation does not hold good, as the reflected pulses from the other end arrive at the hammer before it leaves the string, of which no account is taken in Mr. Das's equation. Experimental determination was made up to the point $l/5$ of the string at an interval of 1 cm. The result is shown graphically in Fig. 1.

Fig. I



The dotted curve gives the theoretical results whilst the continuous curve represents the experimental values. In Figs. II and III are given the curves showing the amplitudes of the octave and the third harmonic for different striking points along the string up to the middle point obtained experimentally. The ordinates are the amplitudes, and the abscissæ are the distances of the striking point from the end.

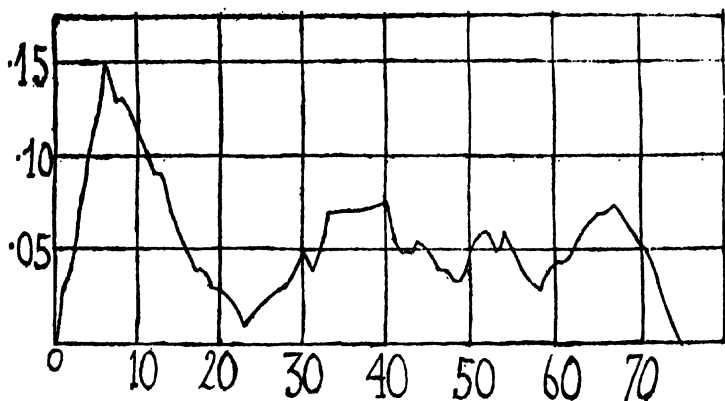


Fig. II

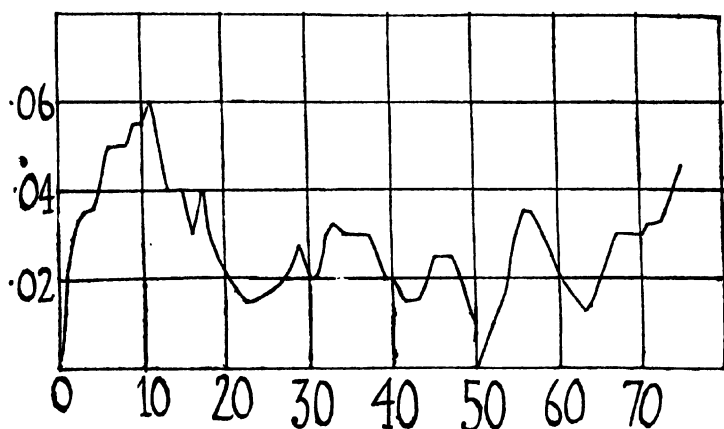


Fig. III

3. Conclusion.

The very close agreement between the observed and the calculated values as seen from the above tables is a very good confirmation of the equation. It is also seen from the above tables that the observed amplitudes are always less than the calculated ones, a fact which follows from theoretical considerations. For, some amount of dissipation of energy takes place in the system of which no account is taken in Mr. Das's theory. Moreover some allowance must be made for the fact that a certain time elapses, however small that may be, between the striking of the hammer and the measurement by the microscope, the effect of which is to lower the observed values.

It is seen from the curve showing the variation of amplitudes of the fundamental with striking point, that near about the point $l/7$ the amplitude of the fundamental is a maximum. Now in an actual pianoforte the choice of striking point is generally made at about the same point, *viz.* $l/7$. This may be due to the fact that the amplitude is a maximum at this point. The reason for this fact is that beyond this point the $\sin nt'$ and $\cos nt'$ terms in the integrals

change sign and so a part of the curve comes on the negative side of the time axis. At this point the value of the duration of contact becomes nearly $\frac{3}{4}$ th of the period of the string. The position of the striking point for which the amplitude of the fundamental is a maximum also depends on the mass of the hammer. As the weight of the hammer increases the point of maximum is shifted towards the end.

The rapid recovery of the amplitude of a harmonic as the striking point is moved away from one of its nodes is markedly shown in the results of the investigation.

The calculations have been made up to the point $l/5$, for beyond this we have to take into account the reflections from both ends which I hope to do in a later paper.

Further investigations as to the duration of contact and the effect of elasticity of the hammer are now in progress and will be given in a later paper.

The investigation here described was carried out in the Palit Laboratory at the University College of Science at the suggestion of Prof. C. V. Raman, Palit Professor of Physics, and the author is indebted to him for the unfailing interest he has taken during the progress of the work.

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1st May, 1923.

VIII. On the Temperature Variation of the Electrical Conductivity of Copper and Iron fused with Mica.

By

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Three years ago, there was published in the Philosophical Magazine¹ a paper by A. L. Williams and Miss Mackey containing the results of some experiments on the electrical conductivity of mixtures of copper and iron with mica. The most remarkable feature about them was the very large variation of conductivity with temperature.

In Part B of the above paper are given the resistances of two samples each of copper-mica and iron-mica mixtures as measured by Miss Mackey. In all these, I find that the variation of resistance with temperature is expressed over the whole range of the experiment with a fair degree of accuracy by means of either of the formulæ

$$R = PT^{-1} e^{Q/T} \quad \dots (1)$$

$$\text{or} \quad R = P_1 T^{-1/2} e^{Q_1/T} \quad \dots (2)$$

where R=resistance of the sample

T=temperature measured on the absolute centigrade scale and P, Q, P₁, Q₁, are constants,

¹ Phil. Mag., Vol. XL, 1920, p. 281.

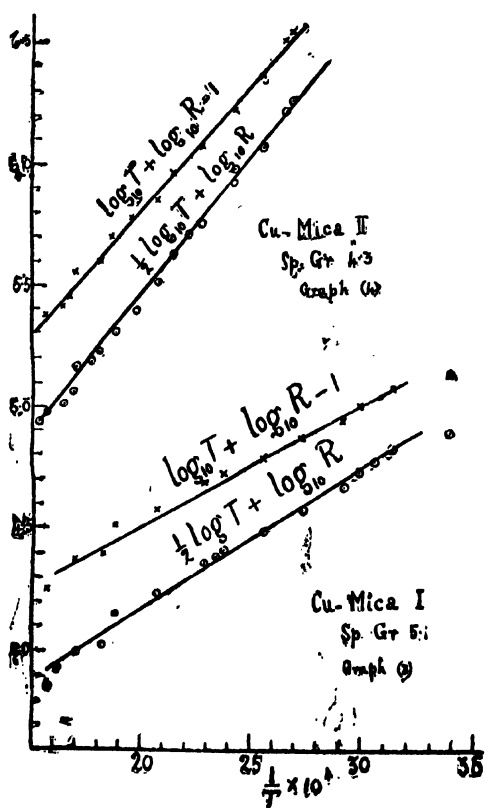
Taking logarithms,

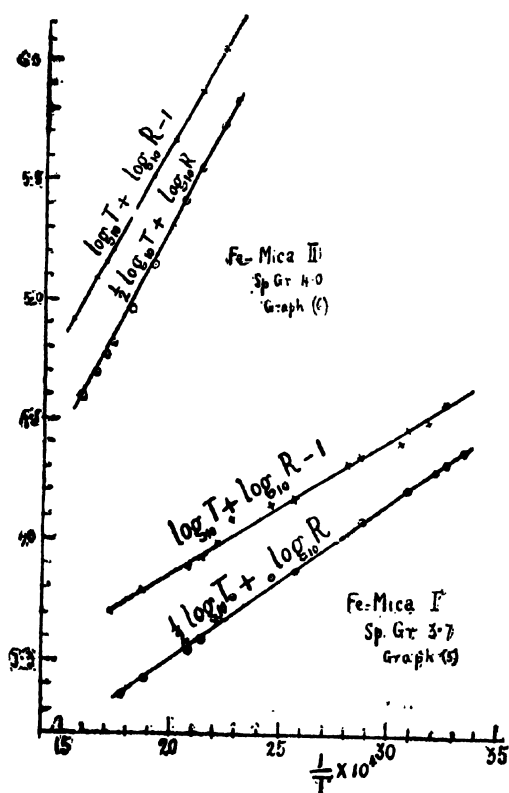
$$\log_{10} R + \log_{10} T = \log_{10} P + \frac{Q}{2 \cdot 303T}$$

and

$$\log_{10} R + \frac{1}{2} \log_{10} T = \log_{10} P_1 + \frac{Q_1}{2 \cdot 303T}$$

The quantities $\log_{10} R + \log_{10} T$ and $\log_{10} R + \frac{1}{2} \log_{10} T$ are plotted against T^{-1} in graphs 3 to 6, and the graphs are straight lines.





(The graphs have been given the same numbers as in Miss Mackey's paper.)

Since the samples used were irregular, their specific resistances could not be determined and there is no significance in calculating the values of P and P_1 . In the following table are given the values of Q and Q_1 for the different specimens :

Specimens.			Sp. gr.	Q	Q_1
Cu-mica I	5.1	1.27×10^3	1.13×10^3
Cu-mica II	4.3	2.63×10^3	2.36×10^3
Fe-mica I	3.7	1.50×10^3	1.30×10^3
Fe-mica II	4.0	4.01×10^3	3.78×10^3

It will be noted that Q and Q_1 show a decrease with increasing metal content in Cu-mica and an increase with increasing metal content in Fe-mica.

Theoretical.

The expression $R = PT^{-1} e^{Q/T}$ can be put into the form $\sigma = P^{-1} T e^{-Q/T}$ where σ is the conductivity of the sample. This is of the same form as the expression for the variation of the number of thermions in equilibrium with a metal in a vacuous enclosure

$$n = A T^{\lambda} e^{-b/T} \quad (1)$$

where λ is a constant which may be given any value between 0 and $\frac{1}{2}$ without appreciable difference (O. W. Richardson:

Emission of Electricity from Hot Bodies, Chap. III). Assuming the concentration of free electrons in the mixture to vary in a similar manner with temperature, we can calculate the variation of conductivity with temperature. On the electron theory of metallic conduction,

$$\sigma \propto \frac{n e^2 l v}{T}$$

where

n = number of free electrons per unit volume

v = root mean square of the velocity of electrons

l = mean free path of the electron

and

e = electronic charge

Since

$$n \propto T^{1/2}$$

$$\sigma \propto n v^2 / T^2$$

If we assume that l is independent of temperature, and n varies with temperature according to equation (3)

$$\sigma \propto T^{\lambda - \frac{1}{2}} e^{-b/T}$$

If $\lambda = 3/2$

$$\sigma \propto T e^{-b/T}$$

and if $\lambda = 1$.

$$\sigma \propto T^{\frac{1}{2}} e^{-b/T}$$

Since writing the above note, there has appeared in the Journal of the Franklin Institute a paper on the "Resistivity of vitreous materials" by L. L. Holladay where the author has put forward the same type of formula to explain the variation of the resistivity of a number of specimens of glass with temperature. It is of interest to compare the values of Q for the glasses and the metal-mica mixtures. The values calculated by Holladay for the glasses range for different specimens from 9.37×10^3 to 1.30×10^4 and are thus of the same order of quantities as those calculated for the metal-mica mixtures.

X. The Diffraction of X-rays in Liquids, Liquid Mixtures, Solutions, Fluid Crystals and Amorphous Solids.

By

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2. Comparison with the optical scattering problem.
3. X-ray scattering at very small angles with the primary beam.
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6. Calculation of the intensity of X-ray scattering.
7. Comparison with observations.
8. Liquid crystals.
9. Amorphous solids.
10. Summary and conclusion.

1. Introduction.

When a narrow pencil of homogeneous X-rays passes through a thin layer of liquid and is received on a photographic plate, it is found that with a sufficient exposure, besides the central spot given by the undeviated pencil, there appears on the plate also a circular diffraction halo surrounding the

¹ A preliminary note in which the theory developed in this paper was indicated appeared in "Nature," Feb. 10, 1923, p. 185.

centre and separated from it by a relatively clear space. This somewhat surprising observation was made by Debye and Scherrer¹ in the course of their work on X-ray diffraction. The same result has also been obtained in some recent X-ray studies by Hewlett² in which the ionization method was employed. Keesom and Smedt³ have also studied the phenomenon in the case of several liquids by the photographic method and find that in some cases there is a weak second halo outside the first and even the suspicion of a third.

The behaviour in this respect of the special class of substances known as liquid crystals and studied by Lehmann and others is obviously of much interest. At the suggestion of Prof. Debye, observations were made by Hückel⁴ with several of these substances, particularly with p-azoxyanisol and cholesteryl-propionate which were studied in (1) the solid crystalline, (2) the liquid crystalline, and (3) the liquid isotropic conditions. In the solid crystalline state, several sharp rings similar to those of other crystal powders were obtained, but remarkably enough, there was no notable difference shown by the photographs obtained with the isotropic and crystalline liquid states. In both cases a single diffraction halo appeared as in the case of ordinary liquids. It may also be mentioned that observations by Freidrich⁵ on the scattering of X-rays by wax and other amorphous solids and by Jauncey⁶ on the scattering by glass similarly show a maximum at a considerable angular distance from the undeviated pencil. It thus appears that in the three cases of an isotropic liquid, of a liquid crystal, and of an amorphous solid, we have essentially similar phenomena exhibited.

¹ *Nachrichten Gottingen*, 1916.

² C. W. Hewlett : *Physical Review*, XX, 1922, p. 688.

³ Keesom and Smedt : *Proc. Roy. Soc. Amsterdam*, XXV, 1922, p. 118, and XXVI, 1923, p. 112.

⁴ Hückel, *Phys. Zeit*, 1921, p. 561.

⁵ Freidrich : *Phys. Zeit*, 14, 1913, p. 317.

⁶ Jauncey : *Phys. Review*, XX, 1922, p. 405.

From the survey of the literature, it would appear that no satisfactory explanation of the appearance of the diffraction halo in these cases has so far been put forward. One view that has been suggested¹ is that the halo might be a diffraction-effect arising from the finite size of the molecule or the co-operation of the different atoms in it. This suggestion however must be negatived in view of Keesom and Smedt's observation that a liquid like argon which presumably has monatomic molecules shows the phenomenon in much the same degree as substances with more complex molecules. Another view that has been put forward by Hewlett² is that ordinary liquids possess something resembling crystal structure. The idea that in a liquid there are large groups of regularly arranged atoms is also put forward by A. H. Compton³ in his recent report on X-ray scattering, when referring to observations by Hewlett and Duane. These hypotheses by Hewlett and Compton appear to us to be somewhat artificial; they have obviously been introduced in order to explain the observed effects, but lack independent justification. Keesom and Smedt have attempted to interpret their results as due to the interference of the effects of two neighbouring molecules, using for this purpose a formula proposed by Ehrenfest. Their theory, however, appears to us inadequate. The essential features of the phenomenon are the region of the relatively very small intensity of scattering surrounding the central spot, and beyond this a moderately sharp diffraction-halo, having a much greater intensity than the scattering at large angles. Neither of these features is indicated by Ehrenfest's formula. To make the point clear we give below in Fig. 1C the curve of intensity for benzene reproduced from Hewlett's paper, and in Fig. 1A for comparison with it a graph of the intensity calculated from Ehrenfest's formula. It will be seen that

¹ Debye: referred to by Hückel.

² *Loc. cit.*

³ Bulletin, National Research Council, U. S. A., No. 20, p. 14.

there is little in common between them. Finally we should mention an attempt which has been made by L. Brillouin¹ to explain the phenomena of X-ray diffraction in liquids and amorphous solids on the basis of the quantum theory of specific heats. We give in Fig. 1B a graph of the intensity in different directions drawn from his final formula. It will be seen that it bears no resemblance whatever to the observed result given in Fig. 1C.

We propose in this paper to approach the problem from an entirely different standpoint. In a series of publications² that have appeared in the course of the last two years, the authors and their co-workers have discussed the optical problem of the scattering of light in liquids under various conditions and shown that the experimental evidence amply confirms the

¹ *Annales de Physique*, Jan-Feb. 1922, pp. 88-122.

² 1. Notes by C. V. Raman in *Nature*, November 10, 1921, and several subsequent issues.

2. "Molecular Diffraction of Light" by C. V. Raman, Calcutta University Press, February, 1922.

3. "The Molecular Scattering of Light in Water and the Colour of the Sea" by C. V. Raman, *Proc. Roy. Soc.*, April 1922, pp. 64-80.

4. "The Molecular Scattering of Light in Vapours and in Liquids and its Relation to the Opalescence observed in the Critical State" by K. R. Ramanathan, *Proc. Roy. Soc.*, Vol. 102, 1922, pp. 151-161.

5. "The Molecular Scattering of Light in N-pentane" by R. Venkateswaran, *Trans Chem. Soc.*, Vol. 121, 1922, p. 2655.

6. "The Molecular Scattering of Light in Liquid Mixtures" by C. V. Raman and K. R. Ramanathan, *Phil. Mag.*, Jan. 1923, pp. 213-224.

7. "The Molecular Scattering and Extinction of Light in Liquids and the Determination of the Avogadro Constant" by C. V. Raman and K. S. Rao, *Phil. Mag.*, March 1922, pp. 635-640.

8. "Electromagnetic Theory of Scattering of Light in Fluids" by K. R. Ramanathan, *Proc. Ind. Assn. for the Cultivation of Science*, Vol. VIII, Part I, pp. 1-22.

9. "The Visual and Photographic Albedo of the Earth," by K. R. Ramanathan, *Astrophysical Journal*, April, 1923.

10. "The Molecular Scattering of Light in Benzene Liquid and Vapour" by K. R. Ramanathan, appearing in the *Physical Review*, 1923.

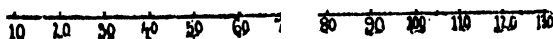
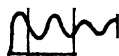
11. "The Molecular Scattering of Light in Liquid Mixtures" by J. C. Kameswar Rao, appearing in the *Physical Review*, 1923.

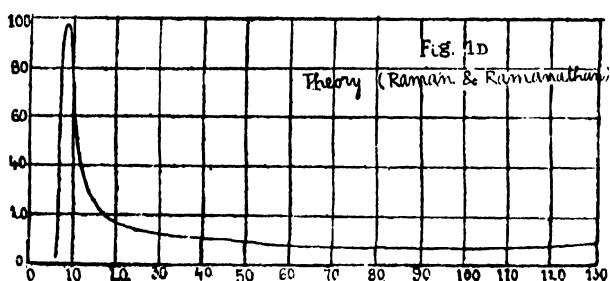
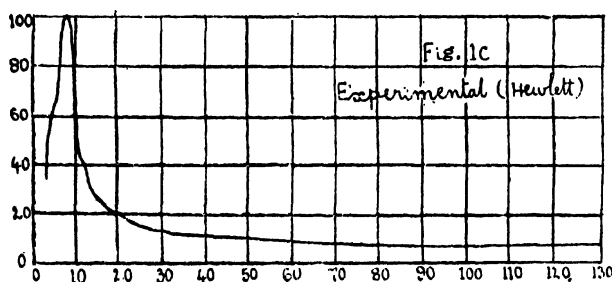
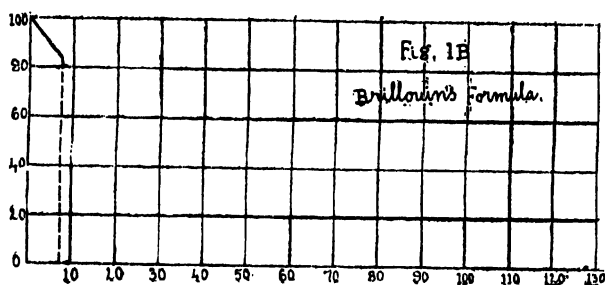
And other forthcoming papers.

statistical-thermodynamical theory of light-scattering developed by Smoluchowski and Einstein. The essential idea of the Einstein-Smoluchowski method is to treat a fluid as a continuous substance subject to local changes of density determined by thermodynamical considerations. Leaving out of account the effects due to the anisotropy of the molecules, the theory leads in the optical case to precisely the same results as those given by a more explicitly molecular treatment. This is due to the circumstance that the length of light waves is vastly greater than the scale of molecular dimensions, and hence the assumption involved in treating the substance as a structureless continuum does not lead to any appreciable error. The case is different however when we deal with the problem of diffraction of X-rays. The wave-length here is less than the average distance apart of the molecules, and in applying the statistical-thermodynamical considerations developed by Smoluchowski and Einstein, we have explicitly to take into account the fact that the medium is not continuous, but consists of a finite number of discrete particles. When this is done, the experimental results are explained quantitatively in a satisfactory manner. Fig. 1D gives the graph of intensity calculated from the formula we have developed in this paper. When account is taken of the imperfect homogeneity of the X-rays used by Hewlett, it will be seen that his experimental curve reproduces with remarkable fidelity the indications of theory.

1. 1A'

Ehrenfest's Formula

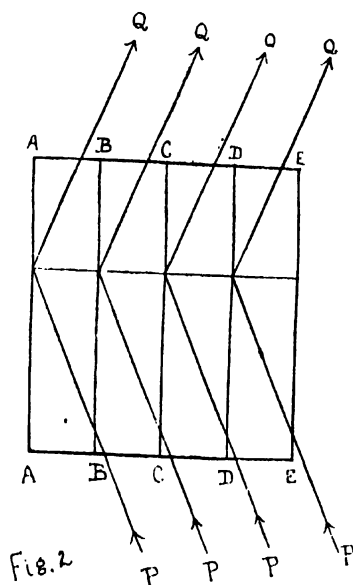




2. Comparison with the optical scattering problem.

In order more clearly to appreciate the relations between the optical and X-ray problems, it is desirable here to give a brief outline of the theory of the former case. In Fig. 2, let ABCDE represent a unit volume of fluid (supposed of refractive index only slightly differing from unity) on which a parallel pencil of light is incident. Let n be the total number of molecules in it and let PPPP represent the in-falling rays and QQQQ the scattered rays in the particular direction under consideration. The volume ABCDE

may be divided into a very large number of slices by a series of equidistant planes perpendicular to the plane of the paper and equally inclined to the incident and scattered rays. It is assumed that each slice is thick enough to be several molecules deep and yet very thin compared with the wave length of the incident light. It is obvious that with these assumptions the scattered waves arising from the molecules in each slice may be taken to be all in identical phases. Let, A, B, C, D, etc. be successive



planes which are situated at such intervals that the path differences of the scattered rays arising from layers adjacent to them differ by one wave-length. Each of the slabs AB, BC, etc., thus contains a considerable number, say r , of the thin slices into which the medium was supposed to be divided. Then $AB = BC = CD = DE = \lambda \cdot 2 \sin \frac{1}{2} \theta$ where θ is the angle of scattering and λ is the wave-length of the incident radiation. Since the effects of all the molecules in a given slice, say the p th, of any one slab are taken to be in the same phase, they also agree in phase with those from the p th slice of every

other slab. The amplitudes of the scattered waves arising from all the molecules lying in the p^{th} slices of all the slabs may then be added up; and their resultant amplitude is proportional to the total number of molecules contained in the slices thus added together; denote this by n_p . The whole scattering may then be found by summing up the effects proportional to $n_1, n_2, \dots, n_p, \dots, n_r$ of the successive slices, due regard being had to their respective phases, which are distributed at regular intervals from 0 to 2π . If $n_1 = n_2 = \dots = n_p = \dots = n_r$, the effects of the different slices would completely extinguish each other by interference. This corresponds to the case of a completely homogeneous medium, that is, a crystal at the absolute zero of temperature for waves of length great compared with its grating constant. In every other case, n_1, n_2 , etc. would show fluctuations in value, and part of the incident energy would appear as scattered or internally reflected radiation. We know that $n_1 + n_2 + \dots + n_p + \dots + n_r = n$ and hence, denoting $n_1 - n/r = \Delta n_1$, $n_2 - n/r = \Delta n_2$, so on, the resulting effect would simply arise from the quantities $\Delta n_1, \Delta n_2$, etc. which represent the fluctuations from the mean density, the part due to the mean density itself disappearing by interference.

We can now consider the magnitude of the fluctuations $\Delta n_1, \Delta n_2$, etc. in different cases. We take first the case of an ideal gas which has been discussed by H. A. Lorentz.¹ Here the distribution of the molecules is a purely random one, and the average expectation of magnitude of the fluctuations $\Delta n_1, \Delta n_2$, etc. can be very simply shewn from probability considerations to be $\sqrt{n/r}$. If further, we make the assumption which is *a priori* justifiable in the case of an ideal gas—that the quantities $\Delta n_1, \Delta n_2$, etc. are as often positive as negative and vary quite independently of each other, then to find their aggregate effect, we add up *not* their amplitudes in their respective phases, but their intensities without regard to phase.

¹ Proc. Roy. Soc. Amsterdam, Vol. 13, 1910, p. 92.

The total scattering will thus be proportional to $r(\sqrt{n/r})$, or simply n , that is, to the total number of molecules in the fluid per unit of volume. This is the well-known Rayleigh law of scattering.

• We next consider the case of a gas not obeying Boyle's law in which the distribution of molecules is no longer a random one. Here, applying Boltzmann's principle of entropy-probability, we find the mean value of Δn_i to be, $n \sqrt{R\beta T/N}$ where R and N are respectively the gas constant and Avogadro constant for a gram-molecule, T is the absolute temperature and β is the isothermal compressibility of the fluid. Assuming as in the case of an ideal gas that Δn_i , Δn_j , may be as often positive as negative, and that their values are quite independent of each other, we get the total scattering by squaring and adding their intensities. The net result is thus proportional to

$$n^2 RT\beta/N \quad \dots (1)$$

and is thus proportional to the compressibility and to the absolute temperature and to the *square* of the density. For a gas obeying Boyle's law, β is the reciprocal of the pressure and it is easily seen that the expression reduces to n , which is the Rayleigh law of scattering.

For a liquid or very dense vapour of which the refractive index is sensibly greater than unity, the discussion proceeds on exactly the same lines as above, except that the local electromagnetic field due to the molecules themselves cannot be neglected in comparison with the field due to the incident wave and must be taken into account as in Lorentz's theory of dispersion. This increases the intensity of the scattered light without affecting its state of polarisation as shown in the paper (8) by Ramanathan quoted above. The scattering due to a unit volume is now proportional to

$$\left(\frac{\mu^2 + 2}{3}\right)^2 n^2 RT\beta/N \quad \dots (2)$$

where μ is the refractive index of the fluid. When μ is sufficiently nearly equal to unity, (2) reduces to (1).

3. *X-ray scattering at very small angles with the primary beam.*

It will be seen that the simple treatment given above depends essentially on the possibility of dividing up the medium into slabs of thickness $\lambda/2\sin\frac{1}{2}\theta$ which can be further subdivided into several slices, each of which is many molecules thick, so that the fluctuations of density in any slice can be assumed to be independent of those in neighbouring ones. This, in general, is obviously possible only when λ is large, which is true in the optical case. When λ is comparable with molecular dimensions, and θ has any moderate value, each slab of thickness $\lambda/2\sin\frac{1}{2}\theta$ would be only a few molecules thick, and it would no longer be possible to assume that when it is sub-divided into thinner slices, the fluctuations in the different slices are uncorrelated, *i.e.*, independent of each other. In fact, it is easy to see that when the volume of the liquid is divided into very thin slices each only a molecule or so in thickness, any excess of density in one slice necessarily involves a deficiency in the adjoining slices and *vice-versâ*. The simple summation of the intensities of the scattered waves due to the density-fluctuations in the different slices, thus ceases to be admissible.

In one case, however, the Einstein-Smoluchowski theory may be applied as it stands to the problem of X-ray scattering. This is when the angle of scattering θ is very small. The thickness $\lambda/2\sin\frac{1}{2}\theta$ of the slabs AB, BC, CD, is then appreciable and may be made as large as we please by sufficiently decreasing θ . For instance if $\lambda = 0.71$ A. U., and $\theta = 10'$ of arc, $\lambda/2\sin\frac{1}{2}\theta = 239$ A. U. and each of the slabs AB, BC, etc., would, if we take the case of benzene liquid, be about 50 molecules thick. This thickness should be ample to enable the Einstein-

Smoluchowski theory to be applied. The isothermal compressibility β being 90×10^{-12} dynes per cm^2 for benzene, it is easily shown by calculation that the scattering given by formula (1) is $1/40$ times smaller than in proportion to the number of molecules per unit volume. The scattering of X-rays at these small angles by liquids is thus almost negligible. Even for an angle of scattering of 2 degrees, a layer $\lambda/2\sin\frac{1}{2}\theta$ thick would be about 5 molecules deep, and though the Einstein-Smoluchowski theory would not be strictly valid, it could still be applied as a rough approximation, and the result indicated, *viz.*, that the scattering is very small would hold good.

It is thus seen to be a simple consequence of thermodynamics that in respect of scattering of X-rays through small angles, ordinary liquids stand in a position not very dissimilar to that of a complete crystal or of a crystal powder. The principal point of difference is that, in crystals, the compressibility is even smaller than in liquids and the scattering at small angles is therefore practically evanescent.

4. Explanation of the X-ray diffraction-halo of liquids.

As the angle of scattering θ is gradually increased, a stage is arrived at when the slab $\lambda/2\sin\frac{1}{2}\theta$ is only one or two molecules thick, and it is clear that the thermodynamical theory based on the idea that the fluid is a structureless continuum must then be modified. The essentially new feature that must be taken into account is that the fluctuations of density in neighbouring slices are no longer uncorrelated. Without going very deeply into the mathematical theory, it is easy to understand in a general way the nature of the results to be expected. For simplicity, we shall consider the case of a liquid which has only a very small compressibility, and in which consequently the thermal fluctuations of density are very small; this means again that the molecules tend to be distributed in space in a manner approaching uniformity,

and not chaotically as in a gas. If, as before, there be n molecules per unit of volume, it is convenient to regard as the mean molecular distance a length λ_0 given by the formula

$$\lambda_0 = kn^{-1/3}$$

where k is a number of the order of magnitude of unity; regarding the exact value of k , we shall have more to say hereafter. Further, let the angle of scattering θ be such that $\lambda = 2\lambda_0 \sin \frac{1}{2}\theta$. Then each of the slabs AB, BC, etc., would on the average be just one molecule thick. It is obvious that in such a case, as has already been remarked, the supposition that when each slab is further subdivided into a number of slices, the effects of the molecules contained in the different slices would practically cut each other out by interference, would be entirely wide of the mark. In the first place, the number of the molecules in the different slices, would show fluctuations of *relatively* considerable magnitude. Further, instead of these fluctuations of density being entirely uncorrelated, they would be almost completely correlated in the direction of amplifying the total observed effect. For assuming that out of the r slices into which the slab AB is divided, the middle slice contains at any instant an excess number of molecules, the chance that at the same instant the slices near the face A or B contain a corresponding deficiency in molecules is very large. Since the scattered waves due to molecules in the middle and the outer faces of the slab differ in path by $\lambda/2$, the effects due to the excess in one slice and the deficiency in the others, would have *identical* phases, and their amplitudes would thus add up. Thus a very large scattering may be expected in the direction referred to, in fact many times greater than if the different molecules were regarded as scattering centres in random distribution of phase.

Theory thus leads us to expect a very large scattering in the direction θ where $\lambda = 2\lambda_0 \sin \frac{1}{2}\theta$, λ_0 being the mean molecular distance. Since as we have seen, the scattering is almost

nothing at small angles, it follows it should increase rather abruptly as θ increases and approaches the value $2 \sin^{-1} \lambda / 2\lambda_0$. On the other hand, when θ reaches and passes this special value, we should expect a fall of intensity which is somewhat less rapid. For, as the angle of scattering is increased, the quantity $2\lambda_0 \sin \frac{1}{2}\theta$ becomes greater than λ , and hence the fluctuations of density in the different slices begin to neutralise each other's effects by interference, but not perfectly, owing to the want of correlation. At large angles, a considerable effect would be left over as the result of this incomplete correlation, and this may be expected to be still much greater than the Einstein scattering obtained in directions nearly parallel to the primary beam.

The theory thus clearly indicates that the diffraction-halo should be fairly sharp at its inner edge, and rather diffuse at its outer margin. These features are well shown in Hewlett's ionisation curves and Huckel's photographs already cited.

5. Analysis of molecular positions in a liquid and in mixtures and solutions.

In order to present the theory of the X-ray diffraction-halo exhibited by liquids as outlined above in a more complete mathematical form, we have to see how the theory of density-fluctuations which is based on the idea that a fluid may be regarded as a continuum should be modified so as to take into account its actual coarse-grained structure. From general thermodynamical considerations, it is clear that the distribution of the molecules in any small volume of liquid can neither be absolutely uniform and geometrically regular and periodic as in a perfect crystal, or absolutely chaotic as in an ideal gas. The character of the distribution as influenced by the thermal agitation and other factors must in fact be intermediate between these two extreme types. The density of matter present must fluctuate

from place to place and these fluctuations of density may be viewed in two rather different aspects. The first way of regarding them is that adopted by Einstein and Smoluchowski, that is, to ignore the independent existence of discrete molecules and confine attention to the total quantity of matter present in volumes which are small enough to be beyond the limit of microscopic observation, but large enough to contain a great number of molecules. This is quite sufficient for the purpose of dealing with the optical problem and also the X-ray scattering at very small angles. The second way of regarding the matter is to take cognisance of the individual molecules and of their movements in order to discuss and analyse the fine structure of the liquid, and this is necessary when we discuss the scattering of X-rays at larger angles. When we consider the fluctuations of density from the first point of view, their magnitude may be predicted completely from a knowledge of the compressibility of the matter in bulk, and it is unnecessary to know either the weight of the molecules or their size and shape. The fine structure of the liquid on the other hand can only be fully determined if we know the properties of the individual molecules. The thermal agitation is *one* of the factors that must be considered in carrying out this analysis of the positions of the molecules in any state of aggregation of matter, but that it is not the only factor is a fairly obvious proposition. To realise this, we have only to recall the extreme case of a crystal at the absolute zero of temperature. Here we have no "thermal" fluctuations of density, but the structure exhibits complex periodic fluctuations of density that do not vary with time.

As a preliminary to the more complete analysis of positions of molecules in a liquid, we shall first set out clearly the theory of density-fluctuations in a liquid in the simpler form sufficient for the optical problem.

Thermal fluctuations of density: Let us assume that the fluid is enclosed in a cube of edge-length L each way in the

fluid, the co-ordinates of any point within this volume lying between the limits

$$0 < x < L$$

$$0 < y < L$$

$$0 < z < L.$$

Let the density of the liquid in any small region be denoted by

$$\rho, \text{ where } \rho = \rho_0 + \Delta$$

ρ_0 being the average density, and Δ the fluctuation. The work done in compressing the fluid contained in any small volume V so that its density is increased by Δ is

$$\frac{1}{2} \cdot \frac{1}{\beta} \left(\frac{\Delta}{\rho_0} \right)^2 V$$

and putting this equal to

$$\frac{1}{2} R T / N$$

we get at once for the mean square of the fluctuation

$$\Delta^2 = \rho_0^2 \cdot R T / \beta N V.$$

The same result may also be derived by assuming that the medium is traversed by plane sound-waves of different wavelengths, whose energy is distributed in accordance with the equipartition principle. Following Einstein,¹ we may write

$$\Delta = \sum_l \sum_m \sum_n B_{lmn} \cos 2\pi l \frac{x}{2L} \cos 2\pi m \frac{y}{2L} \cos 2\pi n \frac{z}{2L}$$

where l, m, n , are positive integers. The potential energy in the sound-wave whose amplitude is B_{lmn} when integrated over the volume L^3 is easily shown to be

$$\frac{L^3}{16} \cdot \frac{B_{lmn}^2}{\rho_0^2} \cdot \frac{1}{\beta}$$

¹ *Annalen der Physik*, 1910, Band 33, p. 1283.

The law of distribution of each B_{lmn} is thus

$$C \exp. \left\{ \frac{-N L^3}{RT} \cdot \frac{B_{lmn}^2}{\rho_0^2} \right\} d B_{lmn}$$

where C is a constant, and it follows that the mean value

$$\frac{1}{N} \overline{B_{lmn}^2} = \rho_0^2 \cdot RT\beta/NL^3$$

which is identical with that given above, since

$$\Delta^2 = \frac{1}{N} \overline{B_{lmn}^2}.$$

This method of analysing the irregular distribution of molecules in a fluid into a system of sound-waves in a continuous medium is of course merely a convenient mathematical artifice. Einstein adopts it in his paper and shows that for each given direction, it is sound-waves of a particular wave-length that are chiefly responsible for the scattering of light; this wave-length λ_1 is connected with the angle of scattering θ and the wave-length of the light λ inside the fluid by the formula

$$\lambda = 2 \lambda_1 \sin \frac{1}{2} \theta$$

The wave-length of the sound waves which are chiefly effective is thus, except for very small angles of scattering, of the same order of magnitude as the wave-length of the incident radiation.

In order that the thermal energy of the fluid may be identified with the energy of propagation of sound waves in it, it must, as is well-known, be assumed that the sound-wave spectrum is limited on the short wave-length side, the smallest permissible wave-length λ_0 being given by the expression

$$\lambda_0 = \frac{1}{2} \cdot \sqrt{\frac{4\pi}{9}} = 1.118.$$

It is thus clear *prima facie* that Einstein's method of considering the problem of scattering must fail when the wave-length of the incident radiation and the direction of observation considered are such that the sound-waves chiefly responsible

for the scattering have a wave-length equal to or less than this limiting wave-length which is determined by the structure of the medium. We have already shown however, that even before this limit is reached, the influence of the discrete structure begins to be felt and the conception of sound-waves in a continuous medium ceases to be appropriate as a method of dealing with the scattering problem.

Analysis of Fine Structure of Liquids: In the foregoing application of the Fourier analysis to the determination of the thermal fluctuations of density, it was tacitly assumed that apart from these fluctuations, the fluid itself could be regarded as a uniform continuum. This limitation must now be dispensed with, and the Fourier analysis applied to the determination of the actual distribution of matter in the fluid. The result of the analysis would depend on the manner in which the molecules, or rather the electrons in them responsible for the scattering of X-rays, are dispersed in space. If they formed a regular space-lattice—(this contingency cannot of course arise in any actual liquid)—the analysis would indicate a definite periodicity in the distribution of matter with wave-length equal to the grating constant of the lattice. Actually, of course, we cannot expect such sharply-defined periodicities or “structural line-spectra” in a liquid. We should rather expect to get as the result of the analysis, a “continuous structural spectrum” having its chief peak of intensity at a wave-length equal to or comparable with the mean distance between neighbouring molecules. We have to find a formula which will indicate the distribution of intensity in the “structural spectrum.” This cannot of course be done completely without a knowledge of the special characteristics of the molecules under discussion. But, by considering only the essential features of the case, it would appear that the problem can, at least approximately, be solved with a knowledge of only the general thermodynamic properties of the fluid.

Let us imagine a cube in the fluid, which is normally of edge-length λ_0 distended or compressed into a cube of edge-length λ_1 ; the work done in the process is given by the expression

$$\frac{1}{2} \cdot \frac{1}{\beta} \cdot \lambda_0^3 \left(1 - \frac{\lambda_1^3}{\lambda_0^3} \right)^2$$

where β is the isothermal compressibility of the fluid. Actually as the result of thermal agitation, the cube might change shape as well as volume. If we take the cube to remain always a rectangular parallelopiped, the three edge-lengths may each be either greater or less than λ_0 . It is only one chance in eight that all the edge-lengths would be *greater* (or less as the case may be) than λ_0 . The average work corresponding to a change of one of the edge-lengths from λ_0 to λ_1 may thus be taken to be

$$\frac{1}{16} \cdot \frac{1}{\beta} \cdot \lambda_0^3 \left(1 - \frac{\lambda_1^3}{\lambda_0^3} \right)^2$$

and its thermodynamic probability may in accordance with Boltzmann's principle be written as

$$\Lambda \cdot \exp - \left\{ \frac{1}{RT\beta} \cdot \lambda_0^3 \left(1 - \frac{\lambda_1^3}{\lambda_0^3} \right)^2 \right\} \cdot d\lambda_1,$$

where Λ is a constant. If λ_1 be taken to represent a wave-length in the Fourier analysis of the distribution of matter in the fluid, the expression just written is the formula for "the distribution of intensity in the structural spectrum." The expression gives a peak at the wave-length $\lambda_1 = \lambda_0$ with intensity falling off more or less rapidly on either side of the peak. It will be understood that here we are dealing with real periodicities in the distribution of matter, and not merely with fictitious mathematical periodicities as in the discussion of the thermal fluctuations of density. Further, these structural waves pass through the fluid in all directions, and

we may more appropriately write as the expression for the intensity in the structural-spectrum,

$$B \cdot \exp. - \left\{ \frac{1}{16} \cdot \frac{N}{RT\beta} \cdot \lambda_0^3 \left(1 - \frac{\lambda_1^3}{\lambda_0^3} \right)^2 \right\} d\lambda, d\Omega$$

where B is another constant, and $d\Omega$ is the elementary solid angle.

The problem is now to determine the wave-length λ_0 of the peak in the "structural spectrum." *Primâ facie*, we shall not be appreciably in error if we take λ_0 to be identical with the mean distance between neighbouring molecules in the fluid. The evaluation of this mean distance is a very important problem in kinetic theory which does not appear as yet to have been adequately discussed. For an ideal gas, Hertz¹ has shown the mean distance between neighbouring

molecules to be $0.554 n^{-\frac{1}{3}}$ where n is the number of molecules per unit of volume. For a regular cubical arrangement of molecules, λ_0 is evidently equal to $1 \cdot n^{-\frac{1}{3}}$, and for the closest² possible packing $\lambda_0 = 2^{\frac{1}{2}} n^{-\frac{1}{3}} = 1.123 n^{-\frac{1}{3}}$. In a liquid, the arrangement of the molecules is intermediate in type between the absolutely chaotic distribution characteristic of an ideal gas and the regular arrangement characteristic of a crystal. Gans³ has attempted to take into account the finite volume of the molecules in the calculation of the mean molecular distance, and found that with increasing density of the fluid, λ_0 increases from $0.554 n^{-\frac{1}{3}}$ to $n^{-\frac{1}{3}}$ and even more. His treatment is however open to certain criticisms, and the numerical values given by him cannot be accepted as correct. The problem is considered afresh in a separate paper by one of us, and the general result emerges that for a liquid, λ_0 is

¹ Math. Annalen 67. 387, 1909.

² Jean's Dynamical Theory of Gases, 3rd Edition, p. 330.

³ Phys. Zeit., XXIII, 1922, p. 109.

$k, n^{-\frac{1}{2}}$ where k is a fraction ranging from about 0.8 to 1.0 according to the nature of the liquid and its condition as to temperature, pressure, etc. λ_0 may also be expressed in terms of the mean linear dimension or diameter σ of the molecule under consideration. The theoretical discussion indicates that in liquids under ordinary conditions λ_0 is of the same order of quantities as σ but may be some 10% to 20% greater.

Liquid Mixtures and Solutions: As we have just seen, the "structural spectrum" of a liquid consisting of only one substance is determined principally by the mean distance between neighbouring molecules and by its compressibility. Passing on to the case of mixtures and solutions, it is not difficult to see that the structural spectrum should, like many other physical characters, be at least roughly an additive property. For, to a first approximation, the volume of a mixture is the sum of the volumes of its components, and hence it is legitimate to assume that the mean distance between two molecules of the same kind in a mixture does not differ notably from what it is in the pure components. Thus if we have a succession of at least three molecules of one kind

A A A

or three molecules of the other kind

B B B

in a line, we have periodicities which are the same as those in the pure components. On the other hand, if we have at least four molecules forming a chain in which the two kinds of molecules alternate,

A B A B

or

B A B A

we would have wave-lengths corresponding to the sum of the two just considered. In the conditions subsisting in a fluid,

the formation of periodic arrays of four or more molecules of this special type is relatively an improbable event, and hence we are justified in assuming that the "structural spectrum" of a mixture or solution would contain principally only those wave-lengths which occur in the pure components. The same reasoning indicates that the distribution of "intensity" in the structural spectrum in the neighbourhood of these principal wave-lengths would be much the same as in the pure components. Hence we may as a first approximation take the structural spectrum of a mixture to be determined by simple addition of the structural spectra of the pure components taken in the proper proportions.

A more exact discussion of the case of mixtures and solutions would involve a consideration of the changes of density and of compressibility which occur when the two substances are mixed, and the influence on the structural spectrum of the local spontaneous fluctuations of density and composition; the precise magnitude of these fluctuations may be determined thermodynamically from the data for the compressibility and partial vapour-pressures of the mixture. *Primâ facie*, the local fluctuations of *composition* of the mixture would have very little influence on the structural spectrum. For, we are only concerned with the average effect corresponding to the mean composition of the whole liquid. If the two components make up the structural spectrum in proportion to their respective concentrations, the average effect would be the same as if the liquid was uniform throughout and exhibited no fluctuations of composition. The fluctuations of *density* on the other hand are all-important, as in the case of a pure liquid. The *compressibility* of the mixture therefore enters in a fundamental way in the problem and where this shows marked deviations from the additive rule,¹ the distribution of

¹ For data regarding the compressibility of mixtures and solutions, see Cohen and Schut's *Piezo-Chemie*, Leipzig, 1919, pp. 113-142.

intensity in the structural spectrum would differ from that given by a simple superposition of the spectra of the two components. Any notable change of volume on mixture may also be expected to result in a shift of the positions of the peaks in the spectra.

6. *Calculation of the Intensity of X-ray Scattering.*

Having analysed the distribution of matter in the fluid into a "Structural spectrum," in other words, into a number of superposed periodic distributions of different wave-lengths, we proceed to determine the X-ray scattering at different angles by this structure. We ignore the periodicities of larger wave-length which may be identified with sound-waves traversing the medium and which, as we have seen, are only of importance when we discuss the scattering in directions nearly identical with the primary beam. For larger angles of scattering, the periodicities of shorter wave-lengths which arise from the discrete structure of the medium are the only ones that need be considered. It is obvious that each of the periodic distributions of matter into which we have analysed the structure of the fluid would cause an internal reflection or enhanced scattering of the incident X-radiation in the direction given by the Bragg formula

$$2 \lambda_1 \sin \frac{1}{2} \theta = \lambda$$

where λ is the wave-length of the incident X-radiation and λ_1 is the wave-length under consideration in the structural spectrum. Since the structural waves traverse the fluid in all directions, the enhanced scattering or internal reflection corresponding to the wave-length λ_1 would occur in all directions coinciding with the generators of the cone of semi-vertical angle θ . Since λ_0 is the wave-length giving the peak of intensity in the structural spectrum, the special value of θ given by the relation

$$2 \lambda_0 \sin \frac{1}{2} \theta = \lambda$$

gives the cone of greatest intensity of the scattered X-rays, and the scattering would be considerably less both for larger and smaller value of θ . The formation of a fairly well-defined circular diffraction-halo in the X-ray scattering by liquids is thus clearly explained on the conception of the structural spectrum.

From the standpoint of the electromagnetic theory, the problem of determining the effect of the periodic distributions of the matter forming the structural spectrum on the propagation of radiation through the substance is very similar to that solved by Einstein in his paper on light-scattering in fluids except that the law of distribution of intensity in the "Structural spectrum" is different from that in the "Sound-wave spectrum." In fact, we can obtain an expression for the scattering due to the "Structural spectrum" merely by a slight modification of Einstein's treatment for the "Sound-wave spectrum." In the optical problem, we have light-scattering of the same intensity in all azimuths when the incident wave is assumed to have its electrical vector perpendicular to the plane of observation. This is due to the fact that the sound-waves of different wave-lengths which, as explained above, are each separately responsible for the scattering in different directions, are all, in accordance with the equipartition principle, of the *same intensity*. In the "Structural spectrum," on the other hand, the periodic distributions of matter of different wave-lengths follow the special exponential law of intensity

$$A \exp. \left\{ -\frac{1}{16} \frac{N}{RT\beta} \lambda_0^3 \left(1 - \frac{\lambda_1^3}{\lambda_0^3} \right)^2 \right\} d\lambda_1$$

Observing, as before, that each periodic distribution of wave-length λ_1 , is responsible for scattering in a specific direction, the distribution of intensity in the diffraction-halo should obviously follow the law of the structural spectrum very

closely. We may therefore write the intensity of the scattered radiation in any direction θ to be simply

$$C_1 \exp. \left\{ -\frac{1}{16} \frac{N}{RT\beta} \cdot \lambda_0^3 \left(1 - \frac{\lambda_1^3}{\lambda_0^3} \right)^2 \right\}$$

where λ_1 is given by the Bragg formula

$$\lambda = 2 \lambda_1 \sin \frac{1}{2} \theta$$

and C_1 is a numerical factor.

A useful verification of the formula is obtained by considering the case of a fluid of great compressibility, *e.g.*, a gas. In such a case, β is very large and the formula indicates, as is otherwise to be expected, that the scattering is of equal intensity in all directions perpendicular to the direction of the electric vector in the incident rays. If the incident X-rays are unpolarised, we should multiply the numerical factor C_1 by $(1 + \cos \theta)$ exactly as in the ordinary theory of light-scattering.

The numerical factor C_1 may be evaluated in the following way. In experiments on X-ray scattering, the wavelength λ is generally much smaller than the mean distance between neighbouring molecules. The concentration of the scattered radiation in the form of a diffraction-halo is due to the arrangement of the molecules not being a random one, and hence there existing a correlation of the phases of the waves scattered by neighbouring molecules,—agreement of phase and increased intensity in certain directions, disagreement of phase and diminished intensity in others. The problem is analogous to that of the diffraction of light by a large number of fine holes arranged in a roughly uniform manner in an opaque screen. We know that in such a case, the integrated intensity of the diffracted light in all directions

together is simply equal to the energy transmitted by any one aperture multiplied by the number of apertures. Exactly the same way, C_1 may be found by integrating the energy of the scattered radiation in all directions and putting it equal to the scattering by one molecule multiplied by the number of molecules in the volume under consideration.

7. Comparison with Observations.

In order to test the indications of the foregoing theory by comparison with experiments on the scattering of monochromatic X-radiation by liquids, we require to know the compressibility β of the liquid and the mean distance λ_0 between neighbouring molecules in the liquid. The latter quantity may be roughly estimated from the known molecular mass M and the density d of the liquid; the best way of finding it is however from the X-ray scattering itself. As is evident, the formula

$$C_1 \text{ exp. } \left\{ -\frac{1}{16} \frac{N}{RT\beta} \lambda_0^3 \left(1 - \frac{\lambda_1^3}{\lambda_0^3} \right)^2 \right\}$$

when graphed gives a strongly pronounced maximum at the wave-length $\lambda_1 = \lambda_0$, and the general shape of the curve reproduces with striking accuracy, the experimental curves obtained by Hewlett by the ionisation method. (See Fig. 1C and Fig. 1D above, in comparing which allowance should be made for the imperfect monochromatism of the X-ray pencil used by Hewlett). From the known wave-length of the X-rays used and from the angle of scattering θ_0 for maximum intensity, λ_0 may be found by using the Bragg formula

$$\lambda = 2\lambda_0 \sin \frac{1}{2}\theta_0$$

That λ_0 thus determined is of the same order of quantities

as the value of the mean molecular distance otherwise found is seen from the following table :

TABLE.

SUBSTANCE.	λ	θ_0	λ_0	$\sqrt{\frac{3}{d}} \bar{M} = n^{-\frac{1}{3}}$	$\frac{c}{\lambda_0 \cdot n^{\frac{1}{3}}}$
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Results of Hewlett.

Benzene	712A	8.5°	4.80A	5.27A	0.911
Octane	„	8.1	5.04A	6.45A	0.781
Mesitylene	„	6.5°	6.28A	6.13A	1.024

Results of Keesom and Smedt—First paper.

Oxygen	1.54A	27°	3.30A	3.59A	0.919
Argon	„	27°	3.30A	3.61A	0.914
Benzene	„	18°	4.92A	5.27A	0.934
Water	„	29°	3.08A	3.09A	0.997
Aethyl alcohol	...	„	22°	4.04A	4.57A	0.884
Aethyl aether	...	„	19°	4.67A	5.55A	0.841
Formic acid	„	24°	3.70A	3.96A	0.934

Results of Keesom and Smedt—Second paper.

Oxygen	712A	12.5	3.27A	3.59A	0.911
Argon	„	13.0	3.15A	3.61A	0.873
Water	„	13.44	3.04A	3.09A	0.984
Nitrogen	„	11.34	3.60A	3.85A	0.935
Carbon disulphide	...	„	13.23	3.09A ?	4.63A ?	0.667

The mean distance λ_0 between neighbouring molecules found from the X-ray data is in every case of the same order of magnitude as $n^{-\frac{1}{3}}$ where n is the number of molecules per unit of volume. The last column gives the value of k where $\lambda_0 \cong k.n^{-\frac{1}{3}}$. It is seen that k is generally about 0.8 or 0.9 as is indicated by theory. It is noteworthy that of the 5 common liquids reported upon by Keesom and Smedt in their first paper, water which has the smallest compressibility has a value of k which is practically unity, while for ether which is highly compressible, k has the relatively low value 0.841.

The compressibilities of the three liquids for which Hewlett has given scattering curves are respectively as follows :—

Benzene	...	90×10^{-12}	dyne/cm ²
Octane	...	120×10^{-12}	"
Mesitylene	...	75×10^{-12}	"

The differences between these are distinctly too small to produce any notable variation in the sharpness of the halo according to the formula. Strictly speaking, however, the halo for octane may be expected to be slightly less sharp than that for benzene or mesitylene. Hewlett's curves seem to indicate that this is actually the case, though owing to the width of the slit used and other complications, the data cannot be regarded as sufficiently precise on the point. Accurate data are as yet not available for any other liquid. It would be of interest to find experimentally whether liquids of high compressibility, *e.g.*, ether, exhibit a more diffuse halo than others, and whether any effect is produced by raising the temperature of the liquid towards the critical point ; with rise of temperature and consequently increase of both T and β , the halo may be expected to broaden, and since λ_0 would increase with rise of temperature, the halo should also approach more closely the direction of the primary rays. The scattering at small angles should also increase in accordance with the Einstein-Smoluchowski formula. These indications of theory remain to be tested by observation.

The preceding calculations are based on the analysis of the fine structure of the liquid into a continuous "structural spectrum" having the mean distance λ_0 between neighbouring molecules as its dominant wave-length. While this analysis no doubt correctly represents the facts in broad outline,¹ it leaves out of account the special features arising from the structural peculiarities of the individual molecules and their influence on the distribution of matter in a closely packed assemblage. Other periods, particularly those with wave-lengths much smaller than λ_0 , may conceivably become prominent when a dense aggregation of matter is analysed. In such a case, fainter diffuse haloes may arise outside the principal one. Then again, anomalies may arise in the case of highly asymmetrical molecules in which, instead of a single value of λ_0 , we may have two or even three separate values of the mean distance depending on the special relative configuration of neighbouring molecules. The principal halo would then itself exhibit a complicated structure which might become better defined at lower temperatures when the thermal agitation and its diffusing influence are minimised.

A convenient way of visualizing the complications that may arise in individual cases is to consider the powder diffraction-haloes obtained by the Debye-Scherrer method with the same substance in the solid crystalline state. It is well-known that with the finest powders in which the individual particles are microscopic or ultra-microscopic crystals, the diffraction-rings obtained by this method are relatively diffuse.¹ If we imagine the process of subdivision of the individual crystals continued gradually, a stage would be reached when the outer rings would practically all have merged into a diffuse general blackening of the photographic film and even the first few intense rings would have broadened out

¹ See, Szigmondy's *Kolloid-Chemie*, Appendix on the X-ray analysis of colloids, by Scherrer.

and merged into a single halo. If we imagine all the crystals broken up into individual molecules, we should still get a halo, because, as we have already seen, thermodynamic considerations ensure a certain degree of uniformity in the spacing of the molecules. In view of this analogy, it may be expected that some of the less prominent details of the powder-halo may also survive and find their counterpart in the liquid-halo, though diffused and modified by the expansion or contraction which takes place on melting.

The considerations indicated above suggest that it would be of great interest to compare the diffraction-halo shown in the liquid state with that shown by the same substance in the state of crystalline powder below its melting temperature. Unfortunately, as yet, sufficient experimental material is not to hand for making such a comparison. In two cases, however, that of water and benzene respectively, the necessary data are available. X-ray powder-photographs of ice have been obtained by Dennison¹ and the results have been discussed theoretically by Sir W. H. Bragg,² who has pointed out a defect in the experimental technique of Dennison's work. The spacings observed in A.U. are chiefly, 3.92, 3.67, 3.44, 2.68, 2.26, 2.07, 1.96, 1.92, 1.53, 1.37, 1.30, 1.17. The first four roughly group about a mean 3.42 A.U. and the next four which are nearly coincident give a strong halo at 2.05 A.U. Allowing for the contraction and re-arrangement which takes place on melting, these spacings are in general agreement with the first and second diffraction haloes found for liquid water by Keesom and Smedt. The X-ray powder photograph of solid crystalline benzene taken by Broome³ shows extremely strong haloes corresponding to spacings of 4.90 and 4.46 A.U., a second prominent group of haloes corresponding respectively to spacings of 3.71,

¹ Physical Review, Jan. 1921.

² Physical Society's Proceedings. 1922, p. 101.

³ Phys. Zeits., March 1923, Plate VI.

3.44 and 3.11 A.U. and a third group corresponding to a spacing of about 2.00 A.U. The first prominent halo of liquid benzene corresponds to a spacing of 4.80 A.U., while Hewlett's curves also show distinct bumps corresponding to spacings of 3.40 A.U. and 2.04 A.U. The agreement appears significant.

Finally, it may be remarked that observations on the X-ray diffraction by liquids consisting of molecules with extended chains of CH_2 groups and the like, *e.g.*, the fatty acids, would be of interest. No data appear to be available regarding these.

Leaving now the case of pure liquids, we may refer in passing to the case of liquid mixtures and of solutions. Wyckoff¹ has studied mixtures of benzene and carbon tetrachloride, water and glycerol, methylene iodide and carbon tetrachloride, and also aqueous solutions of potassium chloride and of alum. The X-ray diffraction-effects shown by the liquid mixtures tried were found by him to be more or less merely superpositions of the effects shown by the components separately. Those due to the aqueous solutions were practically similar to that of pure water. The results for mixtures are in agreement with the approximate theory already indicated. In the case of aqueous solutions, the dissolved material was probably insufficient in quantity to appreciably influence the observed results. Wyckoff has not studied the case of partially miscible liquids. It would be of interest to examine some cases of this kind, special attention being paid to the phenomena observed in the immediate vicinity of the critical solution temperature and for small angles of scattering.

8. *Liquid Crystals.*

Hückel's result, already cited in the introduction, that no notable difference is observable between the diffraction haloes

¹ American Journal of Science, Vol. V, 1923, p. 455.

shown by "turbid" and "clear" anisotropic liquids readily receives explanation in the light of the foregoing theory and of the ideas regarding the constitution of these bodies put forward by Oseen in two recent memoirs.¹ Oseen considers separately two types of interaction between the molecules, (a) forces tending to alter the relative positions of their centres of gravity, and (b) couples tending to alter their relative orientation. The equation of state of the fluid is derived by statistical-thermodynamical considerations on the basis of the assumed laws of interaction between the molecules. Both theory and observation indicate that in the turbid anisotropic liquids, there are regions whose dimensions include many wave-lengths of visible light over which the molecules are (at least approximately) similarly oriented. Similarity of orientation does not however necessarily involve any special regularity of spacing² beyond what may be expected on known thermo-dynamical principles from the compressibility of the liquid. It is probable also that the orientations are not exactly identical but that there is only a mean direction about which they oscillate. The absence of rigidity clearly shows that the definite space-lattices characteristic of solid crystals do not exist in the "turbid" fluids. Since the X-ray pattern is determined by the spacing of the molecules, and since what is observed in the experiments is the aggregate effect of groups oriented in all possible directions, it is clear that the diffraction-halo of a turbid

¹ Stockholm Academy, Handlingar, Band 61, No. 16 and Band 63, No. 1, 1921.

² In this connection, it is of interest to refer to the experiments of Barker (Jour. Chem. Soc., 1906, Vol. 89, p. 1120) and Beilby on the influence of a set of regularly arranged molecules on the crystal formation of an isomorphous substance. Barker found that if a thin film of NaNO_3 be allowed to dry on a polished surface of calcite, the crystals of NaNO_3 had their edges parallel to those of calcite. Beilby has shown that even when there are intervening films of foreign material, the orienting influence was exerted through the films provided they were sufficiently thin. Therefore the existence of an orienting influence does not necessarily connote a definiteness of spacing. (Beilby—Aggregation and Flow in Solids, p. 102.)

anisotropic liquid would differ little from that of the clear liquid.¹ The observations of Hückel are thus readily understood.

A detailed comparison of the X-ray photographs for the solid crystalline, anisotropic liquid, and isotropic fluid states for p-azoxyanisol and cholesterylpropionate reproduced with Hückel's paper is instructive. For both substances, the diameter of the liquid halo is approximately the same as that of the most intense group of rings in the crystal-powder photograph. Further, the haloes for the anisotropic and of isotropic liquids state, though very similar, are not absolutely identical and show slight differences in detail. This is not surprising in view of the fact that the compressibility, and other physical properties depending on the molecular arrangement are not identical for the two states. Further studies in regard to this would be well worth undertaking.

The recent studies of Friedel and Royer² and of Friedel³ on anisotropic liquids with equidistant planes are of great interest in this connection. Friedel characterises as the 'smectic' state an arrangement in which the molecules besides having a common direction are in addition arranged in equidistant parallel layers, and which is thus intermediate between the amorphous and crystalline states of matter. To use the phraseology of our present investigation, the "smectic" state is a state of aggregation for which the "structural spectrum" for a particular direction is similar to that of a crystal, but for perpendicular directions is similar to that of a liquid. The X-ray diffraction by the "smectic" state of matter would thus simultaneously exhibit the characters of a crystal and of a liquid in different

¹ The X-ray method of observation is thus in a sense less powerful than the optical one in this particular field. Detailed studies of the scattering of ordinary light by "turbid" anisotropic fluids in different directions and at different temperatures would be of interest in relation to the varying size of the molecular groups.

² Comptes Rendus, Dec, 1921, June, 1922.

³ Annales de Physique, November, 1922.

directions. The observations of De Broglie and Friedel¹ on the X-ray diffraction by oleate films, and of Piper² and Grindley on the X-ray diffraction by soap-curds may be explained on this basis.

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9. *Amorphous Solids.*

It is well-known that many liquids and liquid mixtures when freed from dust may be cooled much below the ordinary melting temperature without crystallization occurring, and that they then pass into a highly viscous or "glassy" condition.³ The view has therefore gained general acceptance that vitreous or amorphous solids are really super-cooled liquids, the softening temperature being higher than that of observation. We have already seen that even in liquids, the positions of the molecules are not distributed at random but with a certain degree of regularity depending on the compressibility of the substance. Since, by lowering of temperature, the compressibility of a liquid generally diminishes, it follows that when the substance reaches the highly viscous condition, the molecules are arranged with not less than the degree of regularity characteristic of the ordinary fluid condition. The statement frequently made that in an amorphous solid, the molecules are disposed at "random" is therefore certainly erroneous. It is true we do not have that complete regularity of spacing and orientation characteristic of a crystal. Since an amorphous solid is optically isotropic, it follows that the orientation of the molecules does not lie in any particular direction. But the spacing of the molecules has a considerable degree of regularity. The "structural spectrum" of an amorphous solid is therefore very similar to that of a liquid.

¹ Comptes Rendus, March 12, 1913.

² Phys. Soc. of London, Proceedings, August 1923, p. 269.

³ See for instance, G. Tammann, "Aggregat Zustand" Leopold Voss, Leipzig, 1922.

A valuable confirmation of the views expressed above is furnished by observations on the scattering of light in amorphous solids such as optical glass and in supercooled liquids. Observations with liquids such as salol (phenyl salicylate) which may be converted into glassy solids by sufficiently cooling them show that the light-scattering power in the glassy condition remains of the same order of quantities as in the fluid state. The fact that the optical behaviour of an amorphous solid is very similar to that of a liquid is a justification for inferring that in regard to X-ray diffraction as well, they should behave similarly. The observations of Jauncey by the ionisation method referred to in the introduction, and of Wyckoff (1C) show in fact that ordinary glass gives a diffraction-halo very similar to that of a liquid. This does not, as has sometimes been suggested, indicate that glass possesses a rudimentary crystalline structure. The diffraction-halo observed is truly characteristic of the amorphous or non-crystalline condition. The sharpness of the halo is a measure of the regularity in the spacing of the molecules. An extensive series of observations of the X-ray diffraction-halo given by liquids which are supercooled and made to pass into the vitreous condition would be of interest in order further to elucidate the nature of the amorphous condition, and particularly to determine whether, when the temperature is taken below the softening point, any further re-arrangement of the molecules takes place or not.

Incidentally, it may be remarked that the conception of the "structural spectrum" may also be usefully extended to the case of solids and of solid solutions which are not truly amorphous but consist of microscopic or ultra-microscopic crystals packed together. The smaller the crystals, the more diffuse and weaker would be the lines of the "structural spectrum" and the more nearly would the X-ray scattering approximate to that characteristic of a truly amorphous body.

10. *Summary and Conclusion.*

The paper considers the explanation of the diffraction-haloes observed when a pencil of monochromatic X-radiation passes through a film of liquid and is received on a photographic plate. Explanations previously suggested are discussed and are shown to be inadequate.

(1) The explanation of the phenomenon is shown to depend on the consideration that the positions of the molecules in liquids are not at random but possess a certain degree of regularity which can be estimated thermodynamically from the compressibility of the fluid.

(2) The Smoluchowski-Einstein theory of light-scattering in fluids cannot be applied as it stands to the problem of the X-ray scattering owing to the fact that it practically treats the fluid as a continuum, an assumption which is justifiable in the optical case but not in the X-ray problem where the wave-length is much smaller; it is essential here to take into account the discrete structure of the medium.

(3) For very small angles of scattering, however, the Einstein-Smoluchowski theory is applicable even in the X-ray problem, and an explanation is readily forthcoming why liquids scatter very little at such angles.

(4) For larger angles of scattering, the discrete structure of the medium is taken into account by analysing the distribution of matter in the fluid into a continuous "structural spectrum" which has its peak of intensity at a wave-length equal to the mean distance λ_0 between neighbouring molecules. The law of the "structural spectrum" is exponential and is given by

$$A \exp. \left\{ -\frac{1}{16} \frac{N}{RT\beta} \cdot \lambda_0^3 \left(1 - \frac{\lambda_1^3}{\lambda_0^3} \right)^2 \right\} d\lambda,$$

(5) The X-ray scattering in different directions is obtained very simply by combining the law of the structural

spectrum with the Bragg formula,

$$\lambda = 2\lambda_1 \sin \frac{1}{2}\theta$$

and this gives a diffraction-halo with its maximum intensity in the directions for which $\lambda_1 = \lambda_0$. The curve of intensity of the scattering in different directions agrees well with the experimental results of Hewlett, and the mean molecular distance λ_0 with the value deduced from kinetic theory.

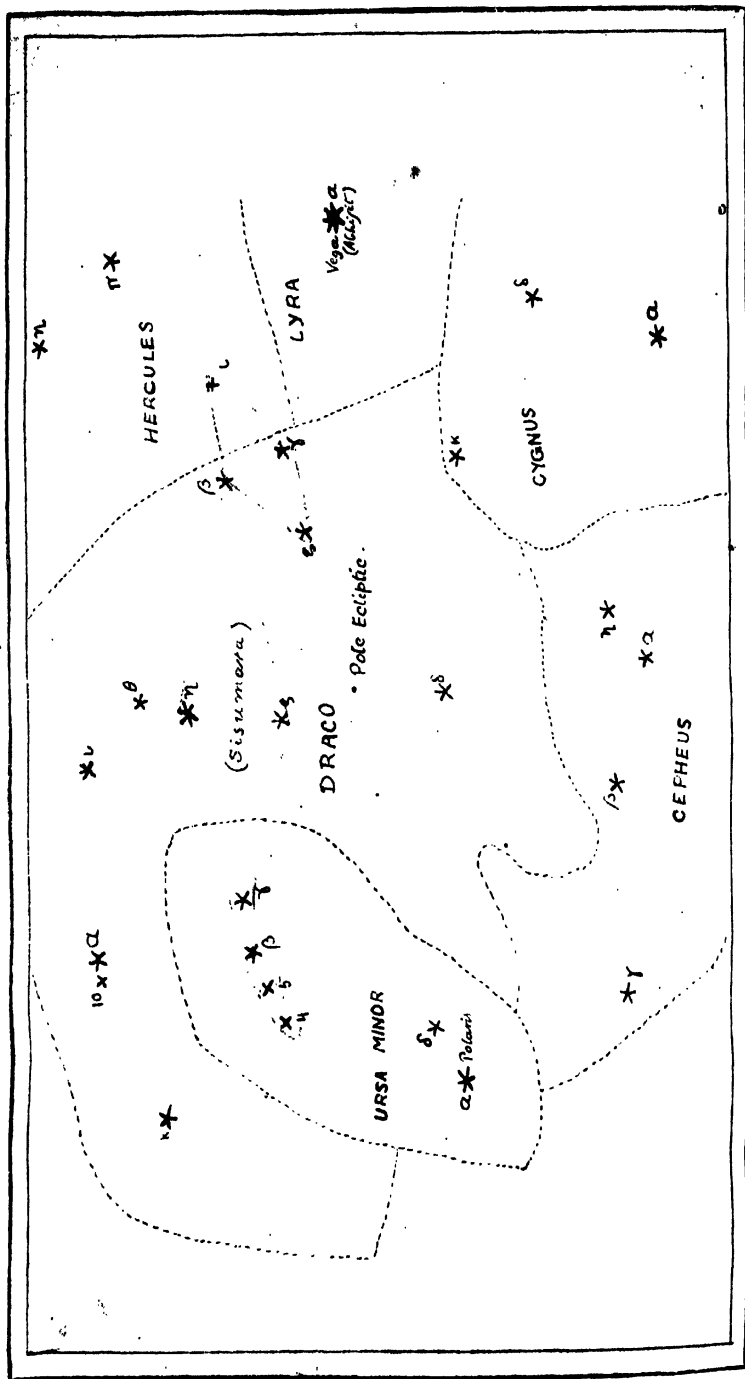
(6) A discussion of the case of liquid mixtures indicates that the X-ray diffraction-halo should be practically a simple superposition of the haloes due to the separate components, as has been observed by Wyckoff.

(7) Diffraction by anisotropic liquids is discussed and the experimental results obtained by Hückel, and by De Broglie and Friedel are explained.

(8) Very similar arguments explain the X-ray diffraction-halo shown by amorphous solids.

(9) Some of the finer details of the halo observed in the experiments are also discussed and are shown to be intelligible in the light of the theory set out.

ASTRONOMY



Map of the Heavens. North polar region.

The Hindu Nakshatras

By

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In my last note on the "Libration of colures and consequent changes in the commencement of the Hindu (Nir-) Ayana year," I made use of the names of the Nakshatras, the identification of which is still to some not yet finally settled. "...He (Al Biruni) speaks very slightly of the practical acquaintance with the heavens possessed by the Hindus of his time, and they certainly have not since improved in this respect; the modern investigators of the same subject and Jones and Colebrooke also complain of the impossibility of obtaining from the native astronomers of India satisfactory identification of the asterisms and their junction stars. The translator in like manner spent much time and effort in the attempt to derive such information from his native assistant but was able to arrive at no results which constitute any valuable addition to those of Colebrooke. It is evident that for centuries past, as at present, the native tradition has been of no decisive authority as regards the position and composition of the groups of stars constituting the asterisms; those must be determined upon the evidence of more ancient data handed down in the astronomical treatises."—Burgess, *Translation of the Surya Siddhanta*, p. 182.

In the following pages, such confirmatory evidence would be put forth as will leave no doubt in anybody's mind regarding the identification of the Nakshatras. With the identification of these, the whole history of the Aryan civilization, as presented in the Vedas, the only ancestral common heritage of us all, will be revealed to us. "...We

have allotted to the Nakshatras more space than to some may seem advisable ; our excuse must be the interest of the history of the system, as part of the ancient history of the use and spread of astronomical science, ... and especially the fact that in and with the asterisms is bound up the whole history of Hindu astronomy.”—Burgess, *Surya Siddhanta*, p. 210.

The word Nakshatra undoubtedly originated from नक्त (night) and त्रे पालने (to preserve)—the dreariness of the night is relieved by the shining and brilliant luminaries in the heavens, or from नाक (heaven) and सत्र (abode) (सत्रिन्—गृहस्वामिन्) the stars having their abode in the heavens. (नाकसत्र—नाक्सत्र—नक्षत्रः—compare अपहरणी—अपहरणी—अवभरणो—भरणी—the asterism Bharani, α, β Triangulum and 41 Arietis—‘यस्मिन् नक्षत्रे प्राणानाम् अपहारम् असुरा अवहन्, तस्य अपभरणी नाम, प्राण भरणात् अपगतत्वात्—तैत्तिरीय ब्राह्मण)। Numerous words in the Vedic literature are derived on the principle ‘परोक्षेण परोक्षप्रिया इव हि देवाः प्रत्यक्षेण प्रत्यक्षहिषाः।’ The plain meaning of which is that gods do not like to call a spade ‘a spade.’

Starting from the Star Revati (ζ Piscium) the ecliptic was divided into twenty-seven equal portions. The names of these in the order of the signs are the following: (1) Aswini, (2) Bharani, (3) Krittika, (4) Rohini, (5) Mrigasira, (6) Ardra, (7) Punarvasu, (8) Pushya, (9) Aslesha, (10) Magha, (11) Purva Phalguni, (12) Uttara Phalguni, (13) Hasta, (14) Chitra, (15) Swati, (16) Visakha, (17) Anuradha, (18) Jyestha, (19) Mula, (20) Purva Ashada, (21) Uttara Ashada, (22) Sravana, (23) Dhanistha, (24) Satabhisa, (25) Purva Bhadrapada, (26) Uttara Bhadrapada, (27) Revati.

The principal stars in each of the above nakshatras (asterisms) will be shortly described. The ancient astronomers first observed their right ascensions and meridian zenith distances and by knowing the latitude of the place

of observation, they got the declinations. Now from the right ascension, the corresponding intercept on the ecliptic was calculated, this was termed the polar longitude (Dhruvaka). From the declination by adding or subtracting the declination of the corresponding point on the ecliptic, they got the polar latitude (Vikshepa). Polar latitude is thus the part of the hour circle cut off between the heavenly body and the point on the ecliptic where the hour circle through the body cuts it. Similarly the polar longitude of a heavenly body is the distance along the ecliptic of the vernal equinoctial point from the point where the hour circle through the body cuts the ecliptic. The true longitude and latitude the ancients were unable to find owing to the absence of any star at the pole of the ecliptic, as rightly remarked by Ranganath, also to the fact of spherical trigonometry not having advanced so far as to calculate true longitudes and latitudes from the right ascensions and declinations of bodies far removed from the ecliptic. Bhaskara in attempting to calculate these put a wrong formula. (Grahaganita, grahachhayadhikara, Sloka 3.) It is well known, the true latitudes being at right angles to the ecliptic, must always be less than the polar latitudes except when the particular body happens to be on the solstitial colure, when they are the same. But Bhaskara's formula gives a value for the true latitude larger than the polar.¹ As a result of this, Muniswara calculated the true latitudes and longitudes of the nakshatras and put wrong figures, making the supposed true latitudes always greater than the polar. Colebrooke

¹ With due respect, I have to show Bhaskara's mistake here. But I have his permission to do so:

परीक्षन् चान्यथा ज्ञयाद् यः परान् न प्रदृषयेत् ।
तस्यैव दृष्यं तस्मिन् न दोषोऽतोऽन्य दृष्यात् ॥

(Siddhanta Siromani. Grahana Vasana, 57 Sloka).

on looking at these wrong figures unsuspectingly doubted that the star Swati (really Arcturus) may be ϵ Bootes.¹

The correction for refraction was unknown to the Hindu astronomers. It is well known, if no correction for refraction is made, the apparent vernal equinox (day and night equal) happens about three days earlier. The apparent autumnal equinox similarly happens about three days after the autumnal equinox. The sun rises or sets on these days not due east or west as understood by us but still to the south of east or west. Hence if we take these as the east or the west points, all the other cardinal points, the position of the pole, the latitude of the place, the position of the equator, will all be more or less changed. If the distance between the apparent and true equinoctial points is always the same, all determinations made from this apparent vernal equinoctial point will not vitiate the results. It must be remembered that the determination of the apparent equinox requires the time-keeper to be sufficiently accurate. The solstices lying midway between the equinoxes will be determined with fair accuracy. The right ascensions of all stars determined from these apparent equinoctial points will show about $2\frac{1}{2}$ degrees more or less in R.A., whereas those determined from the solstitial points will show fairly accurate figures. Owing to these difficulties of practical astronomy several values of polar longitudes and latitudes (calculated from observed right ascensions and meridian zenith distances) show a deviation of about three degrees in ancient determinations. It was not the Hindu astronomers alone that had such difficulties, as will be evident from the following: "Mr. Baily in his new edition of Ptolemy, No. 670, thinks that an error

¹ Colebrooke's Essay on the Indian and Arabian divisions of the Zodiac.

of nearly 3° may have arisen in the latitude of Fomalhaut, from some of the glossators mistaking the numeral $k\gamma'$ for $k\gamma$. To another star λ Orionis he has given latitude ~~vor~~ $16^\circ 30'$, appending the note 'L (iechtenstein) lat. $18^\circ 50'$, but even $16^\circ 30'$ is nearly 3° too great, and there is probably some mistake in all the copies.' (Smyth's Cycle). The fact is, like the Hindus, ancient Greek astronomers also calculated first the polar longitudes and latitudes and thence by applying a wrong formula calculated the supposed true longitudes and latitudes, and hence the trouble. The probable source of error through writers, teachers and learners ('स च लेखक अध्यापक अध्येतदोषैः बहुधा जातः—Bhaskara) is of course common everywhere. For this reason it is extremely difficult for one to verify the ancient figures for the positions of the stars with the help of spherical trigonometry alone. To do so correctly one must determine the apparent equinoctial point for the particular year and thence the cardinal points, etc. From these the positions of the stars should be observed—and then most of the deviations from true figures will be explained. The total horizontal correction of the sun and the moon as given in ancient astronomical treatises (which is wrongly understood to be parallax only) requires verification, taking into consideration all these circumstances. As an example, when the vernal equinox was actually happening about 3 degrees to the east of the star Revati (ζ Piscium), the apparent vernal equinox happening about 3 days earlier was declared to have occurred when the sun entered ζ Piscium. All inferences regarding the time when the vernal equinox was at Revati (ζ Piscium) are consequently wrong by about 200 years (allowing 72 years for the precession of 1°). The star Regulus (Magha) being very bright and of the first magnitude could be observed shortly after sunset. The longitude of this was determined with fair accuracy

(starting from Revati) but not quite correctly because the correction for refraction for the sun while setting was not made. Others determined from the solstitial points, like that of Chitra (Spica) showed fairly correct figures (calculated from true equinox). The ancient figures might also be checked with the help of a precession globe or an armillary sphere by adjusting them properly. With these remarks I shall pass on to the more important consideration of the origin of the Nakshatra names and the most ancient history connected with it.

In describing the Nakshatras, I shall start first from Jyestha (oldest—the most ancient) and Mula (the original). When the vernal equinoctial colure passed through the asterism Jyestha of which the principal star in European astronomy is Antares, it also passed through the asterism Abhijit (Lyra). The star Abhijit (Vega) was then a few degrees distant from the equinoctial colure. In the oldest Vedic literature we have distinct evidence of this phenomenon being observed. The late Mr. Tilak discussed this point and made an inaccurate statement regarding the time when this happened, as had been shown in my “Notes on Indian Astronomy”¹ before. Some time later the star Abhijit was exactly on the equinoctial colure. The Vedic Rishis began to observe the star Vega (Abhijit) ascend, so to say, the polar throne and they very aptly named it Abhijit (from prefix अभि and जि जये to conquer). How glorious was the period when Abhijit became the pole star will be evident from numerous references regarding religious observances connected with this asterism, throughout the Vedic literature. The two asterisms Jyestha and Rohini (the group Hyades) passed through the equinoctial colure during the first period (16000—14500 B. C.) of which we have distinct records. Consequently

¹ Journal of the Department of Letters, Vol. V.

both Rohini and Jyestha were during the ancient Vedic period designated Rohini for both indicated the path of the sun's ascent or descent. "To the earlier observers of the sun's annual motion as measured by his midday elevation these movements indicated the steady ascent of the ruler or the god of the year above the median line which separates the region of his summer glory from the region of wintry imprisonment or burial. It is because of this idea that we find the sun's motion from the time of his passing the equator called his Ascension a term which seems ill chosen if not unmeaning as the astronomer had used it in later times, that is within the last two or three thousand years." (Proctor, *Old and New Astronomy*, p. 117.) The first period as indicated in my 'Libration of Colures, etc.' began when the vernal equinox passed through the asterism Jyestha and the year began about 11 days later when the sun entered Múlá as represented by the two stars λ and ν Scorpionis. Consequently these were termed Múlá (from मूल-origin). The two stars λ and ν Scorpionis are also termed विच्रितौ (Vichritau) in the Vedic literature, Vichritau (from विचृतति सर्वं कुलं हिनस्ति इति—सायण) the two destroyers of ones own race or self—and the identification is simple on a little reflection. The two stars being the stings in the tail of the scorpion, it reminds one of the very ancient belief that scorpions commit suicide on being tortured by fire. 'The belief that scorpions commit suicide on being tortured by fire by stinging themselves to death is of considerable antiquity and is prevalent wherever these animals occur. It is nevertheless quite without foundation in fact; for it has been proved experimentally of late years that the venom has no effect upon the individual itself, nor yet a number of the same species.'¹ Scorpions, however, are

¹ Recent researches have proved these statements to be false. "..... In more than a hundred fights between two scorpions each of the

extraordinarily susceptible to heat and succumb very readily when exposed either to the warmth of a fire or to that of the tropical sun. Moreover when they feel the heat beating upon them they brandish their tails and strive right and left as if to drive off or destroy the unseen enemy ; and there can be no doubt that the belief above alluded to is traceable primarily to observations of the sequence of events just described, the final event being the death of the animal, not however from a self-inflicted wound but from the heat which provoked the behaviour suggestive of suicidal purpose. It might be that under such circumstances a random stroke has now and again wounded the animal itself ; but a wound so inflicted would be accidental, not intentional and at most would contribute in a small measure to the creature's death."—Enc. Brittanica, 10th Ed., 'Scorpion.'

These two stars were several times invoked in the Vedas to heal men from lingering (chronic) constitutional diseases :

‘उदगातां भगवती विचृती नाम तारके ।

वि क्षेत्रियस्य मुञ्चताम् अधमं पाशं उत्तमम् ॥’

(Atharva Veda, 2-2-8.)

‘Arisen are the two blessed stars called the unfasteners (vicrit) : let them unfasten of the kshetriya the lowest, the highest fetters. (The commentator identifies these with Mula, which is the asterism composed of the scorpion's tail)’ —Whitney, *Translation of the Atharva Veda*. Kshetriya is derived from क्षेत्र the system or constitution (‘क्षेत्रं पक्षीशरीरयोः’ —अमरकोष). The two stars' healing virtue would doubtless be connected with the meteorological conditions of the time at which their heliacal rising takes place.’—Whitney

same species whether black, red or yellow, the result was always the same, the one that was stung by its opponent dying almost immediately. 10 seconds being the largest interval between receiving the sting and death.....”—“Scorpions and their venom,” *Nature*. Aug. 1922.

and Burgess.—Surya Siddhanta, p. 337. Now the heliacal rising of a star to cure chronic diseases indicates the approach of the vernal equinox. The heliacal rising of Mula (λ , ν Scorpionis) in Spring occurred about B.C. 16000. This evidence independently proves the extreme ancient date of the Vedas. In later times we find the Arabians ascribing to the star ζ Pegasi the same virtue. ' ζ Pegasi is designated *Iloman* in the Palermo and other catalogues from *Sa'd-al-Iloman*, the hero's happy star, of the Arabians. Tizini called it *Sa'd-al-nu'am*, the ostrich's lucky star; and it is included in the group known as *Su'-udu-l-nujum* the fortunate stars, so named because they appear to the Bedowin Arabs at the dawn of day, on the approach of Spring.' (Smyth's Cycle, Vol. II.) In the Taittiriya Brahmana we find Prajapati the presiding deity of Mula. We also find in the Brahmanas 'संवत्सरः प्रजापतिः।' 'The year is Prajapati' indicating that in ancient times the year began from Mula near about the vernal equinox. Consequently full moon occurring in Nakshatra Mrigashira (Orion), the first month of the year was named Margashira or Agrahayana.

All the literature of the Hindus is replete with references regarding Agrahayana being the first month of the year, which began from Spring. "...संवत्सरोः वा अग्निर्नाचिकेतः । तस्य वसन्तः शिरः । ग्रीष्मो दक्षिणः पक्षः । वर्षापुच्छम् । शरदुत्तरः पक्षः । हेमन्तो मध्यम् ।..." "तै-ब्रा—३-११-१०। Even in Europe the beginning of the year from March to January was very recently introduced. In later times (about B.C. 4000), when the autumnal equinox was passing through Mula the presiding deity of the asterism was changed to 'Nirriti' (calamity).

Anuradha is the asterism preceding Jyestha. The principal star of the asterism is δ Scorpionis, in the forehead of the scorpion. Anuradha is derived from अनु पश्चात् (back) and राधो हिंसायाम् (to bite)—reminding

one that the scorpion bites from the back, there is no danger of being bitten by the mouth. So the star prominent in the forehead, δ Scorpionis, is designated Anuradha.

The Nakshatra preceding Anuradha is Visakha (from वि विशेषेण well—शाखु व्याप्तौ encircled), i.e., the star group which is well encircled by the two pincers of the scorpion, and it is distinctly perceived to be the stars α , β , 20 (γ Scorpionis in German atlases) and ι Librae. The two stars α and β Librae were during the ancient Vedic period termed Visakhe (the two Visakhas). The presiding deity of the asterism is Indraghi (Indra and Agni). This is the only asterism which has got two presiding deities and the reason for which it is comprised in that particular division of the zodiac, Tula (Libra), is to commemorate the coincidence of the vernal equinox and the aphelion (B. C. 14500). The ancients must have observed this phenomenon, to denote the presiding deity of the nakshatra and the name of the rasi (zodiacal constellation) in this manner.

It must be remembered that even in the Vedas, we find mention of five different types of year such as the Vatsara, Samvatsara, Parivatsara, Idavatsara and Idvatsara "...संवत्सराय पर्यायिणीं परिवत्सराय अबिजाताम् इदवत्सराय अतीवरोम् इदवत्सराय अतिष्कदरीं वत्सराय विजर्जरां, etc. (Yajur Veda, Chap. 30, 15th mantra), denoting the tropical, apparent solar, sidereal, lunar and anomalistic years (*vide* also Viswakosha by Nagendra Nath Vasu on 'संवत्सर'); indicating that even in that ancient period the phenomenon of the coincidence of the apsis and the equinox was observed.

With the vernal equinox in Visakhá we have the second period (14500--13000 B.C.) when the year began with the month of Kartika (the moon being full at the beginning of the period in the nakshatra Krittika—Pleiades). We have

copious reference of this period being noticed in the Brahmanas and other Vedic Literature ; Krittika (Pleiades) was then the first of the nakshatras not because the vernal equinox was then in Krittika but because the year began with the full moon in it.

A little consideration of the following quotation will convince any one of the truth of the above statement :
 “देवगृहा वै नक्षत्राणि ।...कृत्तिकाः प्रथमं । विशाखे उत्तमं । तानि देवनक्षत्राणि । अनुराधाः प्रथमं । अपभरणीः उत्तमं । तानि यमनक्षत्राणि । यानि देवनक्षत्राणि तानि दक्षिणेन परियन्ति । यानि यमनक्षत्राणि तानि उत्तरेण ।”—Tait. Brahmana. ‘The nakshatras are the houses of the gods. The nakshatras of the Devas begin with the Krittikas (Pleiades) and end with Visakha (α, β Librae) whereas the nakshatras of Yama (Yama is the presiding deity of the South) begin with Anuradha and end with Apa Bharani.’ So far there is no difficulty and one may as well say that Krittika was first because the vernal equinox was then there. But presently it is stated ‘the nakshatras of the gods move by the south, and the nakshatras of Yama move by the north.’ How is this to be reconciled? The plain and simple meaning is that the nakshatras beginning from Krittika and ending with Visakha were to the south of the equator and those beginning with Anuradha and ending in Bharani were to the north of the equator. The late Mr. Tilak passed over this difficulty without trying to explain it and placed the period 12000 years later. “The Taittiriya Brahmana further states that the Nakshatras of the Devas move towards the south while the nakshatras of Yama move towards the north. The word Dakshina (south) and Uttara (north) are in the instrumental case, and doubts have been entertained as to their exact meaning. But if we accept the statement in the Satapatha Brahmana about....” (Orion, p. 43.) The fact is, after the great Mahabharata war when the vernal equinox was approaching Krittika all

the phenomena were reversed. The Brahmanas as we find them now underwent a revision at that time. Records of both the old and new positions are thus to be found in them. Later on the late Mr. Tilak tried to explain the 'Ekástaká' ceremony. The following points should be noted in this connection : (1) The Ekastaka is the wife of the year, and he (the year) lives in her for that night 'एषा वै संवत्सरस्य पत्नी यद् एकाष्टका । एतस्यां वा एतां रात्रिं वसति ।' (2) Those that sacrifice to Ekastaka sacrifice to the distressed period of the year. 'आर्त्तं वा एते संवत्सरस्य अभिदोक्षन्ते ये एकाष्टकायां दोक्षन्ते ।' (3) It is the season whose name comes last 'अन्त नामानौ ऋतू भवतः ।' (4) It has one fault, *viz.*, that the Visuvan (*i.e.*, equinox or the central day) falls in the rains. 'तस्य एकैव निर्य्या यत् सम्भवे विषुवान् सम्पद्यते ।' Astaka is so named from the eighth day of the latter (bright) half of the lunar month. The lunar months ended then with 'Purnamasi' (full moon), *i.e.*, when the 'masa' or month ended (purna). (*Vide* also 'Taittiriya Samhita' (7-5-6) and the discussion on 'Saptarshi' samvatsara later on.) Hence it was not right for the late Mr. Tilak to say that 'the word Ekastaka is used to denote the eighth day of the latter (dark?) half of the four months.' Now the year has got two portions—one with the sun to the north of the equator and the other to the south of the same. From the autumnal equinox to the vernal one we have the Āsura bhāga—the (dark) portion of the Asuras (night). So that this was really the sacrifice to be performed after the autumnal equinox (रात्रि). We now understand the second and the fourth points 'those that sacrifice to Ekastaka sacrifice to the distressed period of the year,' 'It has one fault, *viz.*, that the Visuvan falls in the rains.' Autumn coming after the rains, it is well known how distressing the rains are especially in Autumn, when cold has already set in, in the northern part of India where the Aryans lived. The

third point will also be clear on a little reflection : ' It is the season whose name comes last.' It is not the last season of the year (the year starting from Spring—Vasanta) but the worst. It is well known to Vedic scholars that 'sarad' (Autumn) being the season when lots of people die (sarad—from शृ शृणाति to kill), the Vedic Rishis implored to live up to hundred sarads Autumns), "...तच्चक्षुर्देवहितं पुरस्तात् शुक्रम् उच्चरत् । पश्येम शरदः शतं जीवेम शरदः शतं शृणुयाम शरदः शतं प्रब्रूयाम शरदः शतं अदीनाः स्वाम शरदः शतं भूयश्च शरदः शतात् । (Yajur Veda, 36-24) so much so that 'sarad' came to mean 'year.' It was to the third and the fourth periods when the year began with the months of Aswina and Bhadra (*ride* my Notes on Indian astronomy—on Libration of Colures, etc.) to which these observances refer. During the third period (13000—11600 B. C.) the month of Chaitra indicated the autumnal equinox. In the fourth period (11600—9600 B. C.) the month of Phalguna indicated the same. The Ekastaka sacrifice was being performed during these months at that remote period. In the fifth period (9600—6600 B. C.) it was being performed in the month of Magha. Later on when owing to the effects of precession half a revolution was completed and seasons were occurring in quite the contrary months, there was much confusion as to when sacrifices should be performed. The earlier writers mentioned that this or that sacrifice should be performed in the month when the moon will be full in this or that nakshatra (asterism). Now arose the 'Mimansakas' whose sole object was to effect a compromise (मीमांसा) between diversified statements and to show the people the right way. How far they succeeded in this—the respective scholars will decide. If now the discussion on the Ekastaka ceremony in Tilak's Orion be gone through one will evidently come to the conclusion that this refers to the period when Chaitra, Phalguna or

Magha in succession indicated the autumnal equinox ; and then the date of the Brahmanas will be carried back by 12000 years. The Ekastaka ceremony in a slightly modified form is still being performed in India in the shape of the Saradiya (autumnal) Puja—Durga Puja, (Durga-Dakshayani—daughter of Daksha Prajapati—wife of Siva Pasupati, Ekastaka wife of Samvatsara, year or Prajapati ‘संवत्सरो वै प्रजापतिः ।’)

Next we come to the nakshatra Swati (Swati-Svati-Svoote-Bootes). The name was derived from श्वन् (dog) and root अत सातत्यगमने (constantly going with), referring to one going with the dog and is evidently Arcturus (a Bootes), holding Canes Venaticae (*Jagdhunde*) in his hand. The श was changed to स like many Sanskrit words in which the two S's are interchanged, e.g., सर्व—शर्व, शस्य—सस्य, वशिष्ठ—वसिष्ठ, शायक—सायक, etc. The great German astronomer Hevelius introduced the two hounds in Europe in 1690. ‘They were given as attributes to Bootes, and the cords which held them passed into his upraised hand. The figure was adopted from elsewhere by the Greeks, no doubt, since they give no certain account of its origin ; their stories making a yaw between Icarus, the father of Virgo of the Zodiac and the farming son of Calisto.’ (Smyth’s Cycle.) The origin of the name of the constellation is to be found in the Vedas : ‘यः रौहिणं अस्फुरत् वज्रबाहुः द्यां आरोहन्तम् ।’ Rig. Veda—II-12-1. ‘Who (Indra) strong-armed rent into pieces the Rauhina (ladder) for scaling the heavens.’ The story is given in the Brahmanas : “There were Asuras the Kalakanjas. They constructed a fire altar with a view to gaining the world of heaven. They, every man of them, put a brick to it. Indra passing himself off for a Brahmin put a brick on for himself saying, ‘This one Chitra (the star Spica) by name is for me.’ They climbed up to heaven. Indra, however, pulled out his brick, and they tumbled

down and became spiders.¹ Two of them flew up to (heaven) and they became the two heavenly hounds.” (Kali Nath Mukherjee’s ‘Popular Hindu Astronomy.’) The plain meaning is that even before 16000 B. C. Jyestha, Mula and other asterisms were below the equator. Hence they were in the portion of the Asuras. Now when the vernal equinox was happening at about Spica (Chitra) they ascended above the equator. Later on when the vernal equinox had receded more than one sign from Chitra those asterisms were again going down. The *rauhina* (ladder) for the sun’s ascent was built with Chitra—indicating clearly that the vernal equinox was passing through it. Formerly the star Swati (Arcturus) was coming to the meridian after Chitra (Spica). But at the time when the vernal equinox was approaching Spica it was observed by the Rishis that Swati was coming to the meridian before Chitra—an unpardonable phenomenon. So they named it ‘निष्ठा’ (from निर्-निर्गत out, and suffix त्वप्) the ‘outcaste.’

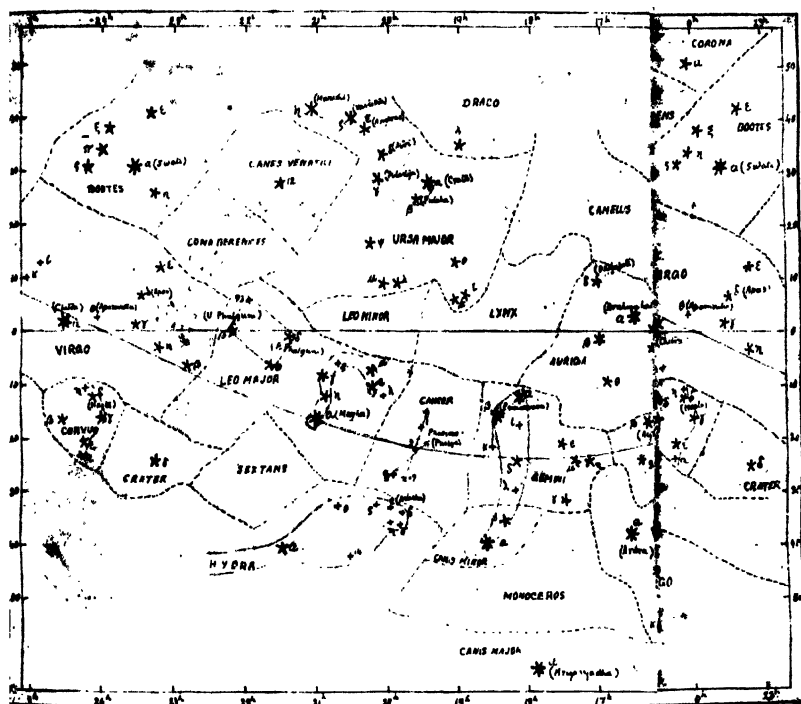
Next we come to Chitra (chitra—brilliant), referring to the bright star Spica. This has been already described.

The star Apam Vatsa (or Apam Napat of the Vedas) as its name implies is a small star—smaller than the star Apas. These two have been rightly identified as θ and δ Virginis. The star Apam Vatsa (θ Virginis) is a small star of the fourth magnitude and none could guess the cause of the same being mentioned in ancient treatises. The reason is simple. When Abhijit (α Lyrae-Vega) was nearest the pole of the heavens, the equinoctial colure passed through the stars θ Virginis (Apamvatsa) and δ and γ Corvus (Hasta). Sometime later when the same was passing through Arcturus (Swati)—a remarkable phenomenon—it also passed through δ (Apas)

¹ This refers to the nebulous region in Virgo and Coma Berenices.

and γ Virginis. The reason of the mention of Apas and Apamvatsa is to perpetuate very ancient notable positions, such as when Abhijit was the pole star and when Swati passed through the equinoctial colure.

Now we come to the more remarkable period, the one beginning with the month of Aswina. During this period the star Abhijit was as near the pole as possible and hence the pole star. About Abhijit becoming the pole star in future Sir John Herschel remarks in his *Outlines of Astronomy* '...about twelve thousand years from now Vega will occupy the remarkable situation of the pole star of the period'; and Professor Young says 'the circle passes not very far from Vega on the opposite side from the present pole star, so that about twelve thousand years from now Vega (α Lyrae) will be the pole star—a splendid one but rather inconveniently far from the pole' (*Manual of Astronomy* p. 147.) The star Agastya (Canopus) was also the southern pole star of the same period (though a little far from the pole—about seven degrees). The frequent reference to Abhijit and Agastya throughout the ancient literature of the Hindus is traceable to this very ancient remarkable situation. Agastya (Canopus) went to the south and disappeared from the view of people living about latitude 24° N in B. C. 17000 and reappeared before the people of the same latitude in B. C. 6000 (about). It is still visible to us but is slowly going to the south. 'Argo is one of the old 48 constellations occupying a very long space in the Southern Hemisphere, but its lucida, Canopus, as well as most of its more important stars, is always hidden from us. There are sound reasons for assigning the highest antiquity to this grand asterism, though the etymologists are crowding on when they derive the English word canopy or covering, from Canopus, as hath lately been imprinted.' (Smyth's *Cycle*, Vol. II, on 'Argo Navis.') We find numerous hymns



in the Rig Veda wherein distinct trace of this very ancient period (the third period beginning with Aswina) is noticed. The chart of the heavens depicting the positions of the important star groups in that period is appended herewith for reference. During this period the vernal equinox was happening about Chitra. There are numerous hymns in the Rig Veda addressed to Usha. Usha is derived from the root उष् दाहे (heat). Usha indicates Dawn when owing to the rising of the sun all beings on earth begin to feel warmth and the dawn in Spring is the most longed for after the cold wintry months. In several hymns 'Usha' is addressed as चित्रा-मघा, अर्जुनी, मघीनी, etc.¹ Yaska simply mentions चित्रा-मघा, अर्जुनी as synonyms for Usha. Why should these be mentioned in addressing Usha nobody guessed so long. Arjuni is another name of the asterism Phalguni (δ, θ, β , 93 Leonis), and Magha is Regulus (with other stars in the group). The simple explanation is that Chitra, Phalguni and Magha rising heliacally in that period indicated the approach of Spring. The Vedic bards on the sight of those important groups in dawn synchronising the advent of Spring, hailed Usha (Dawn in Spring) by the same names. The naming of Usha as Chitra Magha, Maghoni (one having Magha) and Arjuni (asterism Phalguni) is now evident. That Magha was associated with dawn in Spring will be evident on consideration of the derivation of the word from मा (no) and अघ (sin—disease)—without disease. “मघा नक्षत्रस्यैव निष्पाप मूर्त्तयो ‘अनघाः’ ‘अगदाः’ रोग-रहिताः मूर्त्तयः।” Note by S. Samasrami on Tait. Brah., canto 3-1-4. “पितृभ्यः स्वाहा मघाभ्यः स्वाहा । अनघाभ्यः स्वाहा । अगदाभ्यः स्वाहा । etc.” Magha was doing the same function as did ‘Mula’ (Vichritau) before. Another event deserves special notice in this period. The group Orion has

¹ विश्वसिद्धिं प्राप्नुयन् जीवन् त्वे विश्वदूक्षसि सूनरि । सा नो रथेन ब्रह्मता विभावति
अधि विचामये हव° ॥ etc., Rig Veda, 1, 48-10.

by this time gone considerably to the south, owing to the effects of precession. Later on when winter solstice passed through this asterism (about B. C. 10000), it had gone so much to the south that from latitude 40°N and upwards, Orion's belt and the portion below it with Sirius, was not visible at all—only the head (ϕ^1 , ϕ^* and λ Orionis) was visible. Later on it was found to rise again slowly to view, Sirius following. It was during this period that the Vrishakapi hymns were sung in the Vedas. Vrishakapi is derived from Vrisha (bull-Taurus) and kapi (from कम्प to shiver)—a tremulous bull. The asterism Orion is the nakshatra Mriga of the same time. Mriga really means something that is hunted after (मृग-अन्वेषणे to search). Later on it meant animals, that were being hunted down by the hunters (compare also मृगेन्द्र lord of animals that are hunted after—a lion)—antelopes, tigers, lions, wild bulls, etc. Mrigashiras (head of the Mriga) really meant the head of the bull. The reason why the asterism is named Mrigasiras and in ancient western charts Taurus (bull) is shown only by the head is, that during that ancient period only the head of the bull (Taurus—Mriga or Vrishakapi— ϕ^1 , ϕ^* and λ Orionis) was visible from latitude about 40°N . 'Further, as Warren Hastings has remarked, when he was Governor-General of India, 'there are immemorial traditions prevalent among the Hindus that they originally came from a region situated in 40° of North latitude.'—Brennand, Hindu Astronomy, p. 3.

So that Taurus of the Westerners, Vrisha of the Hindus (one of the twelve constellations of the zodiac) must be identified with Orion ;—Hyades must be identified with the Cow Rohini (Rohini is a synonym of cow—*vide* Amarakosha) which was followed by the bull in the Vedic legend. The simple meaning is that when Orion was again being visible it seemed as if Orion is following Hyades, Canis Major (Sirius) also following both. Sirius

(Mrigavyadha—the hunter of the Mriga) shot an arrow which pierced through both the bull—Mriga (Orion) and the cow—Rohini (Hyades—prominent star Aldebaran). So that the real hunter is not Orion but the asterisms Canis Major and Canis Minor. “Ovid indeed throws a doubt upon the gender of this sign (Taurus—represented by Hyades) by making it the transformation of Io....The classical astronomers are, however, very weak in their mythological derivations and zodiacal origins.’... ‘...and it (Canis Minor, Procyon) was popularly considered as Orion’s second hand, or Canicula, which title Horace, Pliny, Hyginus and Gaius support against Germanicus, Julius Firmicus and Appian, who are all for Sirius.’—Smyth, Cycle, Vol. II. That Procyon is really the hunter’s second hand holding a hunting horn composed of the stars α , β , γ , δ , ϵ Geminorum and β and α Canis Minoris will be evident to the observer of the heavens.

“There is the whole story illustrated in the sky; the innocent and the lovely Rohini (Aldebaran), infamous Prajapati (Orion) in full career after her, but laid sprawling by the three jointed arrow (the belt of Orion) which shot from the hand of the near avenger (Sirius) is even now to be seen sticking in his body. With this tale coming down to us from the first period of the nakshatras in India, who could have the least doubt of its persistent identity from the earliest times to the latest.”—Whitney, Essay on Hindu and Chinese systems of Asterisms, p. 53 (quoted from ‘Orion,’ p. 102). Now a reference to Tilak’s Orion, Chapter VII, pp. 157-197 on Ribhus and Vrishakapi will be most interesting. When the vernal equinoctial colure passed through the group Orion (this was an extremely ancient date—B. C. 30000), sacrifices were offered to Vrishakapi (this was allowed by Indra because of the vernal equinoctial colure passing through it). But later on, the vernal equinox having retreated a long way—

Vrishakapi rose on the equator (about B.C. 23000) and people were still sacrificing to Vrishakapi. Indrani got enraged for Vrishakapi having come up to the equator and people still sacrificing to him and not to Indra the sun god in the vernal equinox, to whom really all sacrifices are due. Later on Vrishakapi was again going slowly to the south as if by the orders of Indra and was almost invisible with the exception of the head about B. C. 10000. Later on still, it was slowly rising (at present it has coincided again with the equator) and at this time the hymns to Vrishakapi were sung. Here the description of the phenomenon is exquisite :

‘O Vrishakapi go to the house—the celestial sphere which is cut off and which contains some (unknown) yojanas or stages. From your nether house come to our house. Indra is in the upper (portion) of the universe. O Vrishakapi, you the destroyer of sleep, who are going to the house, come back again, again by your way. We would perform the sacrifices. Indra is....’ ‘O Mighty Vrishakapi! when you rising upwards (or rather northwards) would come to (our) house, where would that great sinner Mriga be? Where he, who misleads people would go? Indra is....’ The explanation of the phenomenon when Vrishakapi returns in his upward march to the house of Indra, the impertinent (sinner) Mriga is not to be seen is that the vernal equinox passing through the group at that time, Mriga (Orion) is not to be seen at night, being in company of the sun. This explains Tilak’s difficulty ‘...For whichever explanation we adopt the question still remains why is the Mriga invisible when both Indra and Vrishakapi are together?’ Orion, p. 192 (note). This Vrishakapi hymn takes us back to the period even beyond 16000 B.C. ‘...and the latest attempt of the kind is that of Pischel and Geldner in their Vedic Studies, Vol. VII, part I. These scholars hold that the hymn narrated a legend current

in old times. In other words they take it, and I think rightly, to be a historic hymn...Pischel and Geldner understand the hymn to mean that Vrishakapi went down to the south and again returned to the house of Indra....' (Orion, p. 176.) The Vrishakapi hymns will be clearly understood in the light of the astronomical phenomenon—the precession of equinoxes in changing the apparent positions of the asterisms. 'Great changes have taken place also in the apparent positions of the constellations in the sky. Six thousand years ago the Southern Cross was visible in England and Germany and Cetus never rose above the horizon.' (Young, 'Manual of Astronomy,' p. 147.) During this period with Vega (Abhijit) and Canopus (Agastya) as the two pole stars, the summer solstice was occurring when the sun entered Sagittarius (Dhanus). Rains set in on the sun's entrance into Sravana (Aquila) and in Sravistha (Delphinus) it was raining copiously. The first was named अवन्या which was another form of स्रवणा (from सु क्षरणे to trickle down) and अविष्टा was स्रविष्टा (अतिशयेन स्रवन्ती too much down pour)—indicating the time of heavy rains on the sun's entrance into this portion. The change of 'S' was done as in lots of other words (e.g., परिस्रुतः-परिभ्रुतः, शृगाल-सृगाल, सार-शार बाषाढ-बाषाढ, etc.) as shown before. The Hindu sign Makara represents the figure of an aquatic animal with its head like that of an elephant. The head of the animal is nicely formed out of the stars α , ϵ , β , δ , μ , λ , γ , Grus and ι , θ , η , 15, μ Piscis Australis. This figure was on the equator during the second and third periods (14500—11600 B.C.). It was then assumed that the rains are due to the heavenly Makara (aquatic elephant, मकर from मुखं किरति—व्यादानं करोति animal with yawning jaws) ejecting water out of its nostrils on the sun's entrance into this sign. Subsequently when this figure had gone considerably to the south not to be visible from northern latitudes (about

2000 B. C.) the figure of Makara was rejected and in its place the constellation Capricornus (goat) was installed by the westerners.

Now we come to nakshatra Hasta (the hand)—from its resemblance to the figure of a hand having the stars η , δ , γ , ϵ , α Corvus for the fingers and β Corvus for the gem worn in the wrist. The star Hasta is identified with δ Corvus. It is well known when the fingers are closed the hand forms a nice bowl (cup—Crater). Evidently the naming of the two constellations Crater (bowl) and Corvus (crow) had been interchanged.

The two nakshatras Purva (former) and Uttara (latter) Phalguni comprising the stars δ , θ , β and 93 Leonis are in the form of a rectangle. The corrupt form of this word Phalguna we still have in 'Palanquin' or Palkee which the four stars above named closely resemble. Phalguni had another name in the ancient Vedic literature—Arjuni. The four stars with the nebulous region in Virgo and Coma Berenices (Berenices Bush) resemble two Arjuna trees with shooting branches and leaves. That Phalguna really meant a Palanquin will be evident from the hymn in the Rig Veda (10th Mandala) where the marriage of Suryá is described.

‘सूर्याया वहतुः प्रागात् सवितायम् अवासृजत् । अवासु हन्वन्ते-
गावोऽर्जुन्योः परिउद्यते । 10-85-13.

Here अवा means the group 'Magha' and अर्जुनी the two Phalgunis as interpreted by Sayana. Astronomically it means, when the sun enters 'Magha' the rays (गावः) of the sun are almost powerless indicating winter solstice (B. C. 15000) and on his entrance into Phalguni they are again revived—symbolically representing सूर्या being carried to her husband's house as if in a palanquin formed of the Phalguni nakshatras. The seventh case ending here is in place of the instrumental. *Vide*

Panini—‘नक्षत्रे च लुपि-मूलेन आवाहयेद् देवीं श्रवणेन विसर्जयेत् । मूले श्रवणे इति वा, so अर्जुन्योः परि उह्यते means, by means of Arjuni (Phalguni-a palanquin) she is carried : *vide* also Tait. Br. Anuvaka I “अर्थ्यन्नः पूर्वं फल्गुणो । जाया परस्तात् ऋषभो भवस्तात् । भगव्य उत्तरे वहतवः परस्तात् वहमाना भवस्तात् ।”¹

The two stars δ , θ Leonis are the Purva Phalgunis and β and η are the Uttara Phalgunis. The presiding deity of the two former is Aryaman (one of the Pitris-gods of the South) and that of the latter Vaga, from which we have Varga (Virgo). Kanya (Virgo) is derived from कान्-दीप्तौ shining, clearly referring to the star Chitra (Spica). By degrees the word Kanya came to denote the virgins for the effulgence all around their body.

The sign Sinha (Lion) was so named from the asterism Magha resembling an instrument of destruction (सिंह from हिंसा). Subsequently Sinha came to mean a lion, the destroyer of other animals.

We come then to the asterism Aslesha. Aslesha is derived from न (no) and श्लिष (आलिङ्गने embrace). That is ‘one with whom there should be no embrace’—evidently, referring to a snake (Hydra) which the asterism resembles. It may also be noted that the snake sheds its skin which closely ‘embraced’ its body. The reason why Aslesha is called a Sarpa (Snake-Hydra) nakshatra is now clear. The star ϵ Hydrae is the star Aslesha.

Let us now come to nakshatra Pushya. The real word was पुष्या and not पुष्या । Any one going through the two texts in Thibaut and Sudhakara Dvivedi’s edition of the Pancha Siddhantika will be convinced of the possibility of this interchange. This clearly indicates the

¹ The asterism Magha resembles a Khadga (sickle with the westerners) used in slaying animals. So it was allegorically described as “the bulls (गावः) are slain (हन्ते) by Agha” (अघासु-अघया ‘नक्षत्रे च लुपि,’ etc. as above, by nakshatra Magha resembling a khadga), presumably for the feast in Surya’s marriage ceremony.

cluster Praesepe which has a nice resemblance to a bunch of flowers. Pushya is also called Tishya (तृष्यन्ति देवा यस्मिन् इति), pleasing to the gods, clearly indicating flowers. In Southern India, Pushya is still identified with Praesepe. If δ Cancri be taken as Pushya it falls in the portion of, Aslesha, starting from ζ Piscium. In all ancient representations this asterism is depicted as an arrow. The arrow is formed of the stars α , δ Cancri with Praesepe as the feathering of the shaft: another shaft is formed of the stars γ , δ Cancri with ϵ Cancri as the feathering of the arrow. The two arrows placed crosswise resemble a compass. That karkata originally meant an arrow will be evident from its derivation (karkata from कर्क-गल or स्कन्ध neck or shoulder and चर to exist—resting on one's shoulder—evidently the arrows). In astronomical treatises karkata means a compass 'त्रिभुज्यका समित कर्कटेन, कक्षाख्यवृत्तं प्रथमं विलिख्य'...Bhaskara. Karkata came to signify a crab (Cancer) from its walking on the strength of the pincers coming out of the shoulder. So the sign Karkata (Cancer) was so named, for the stars resemble a compass.

The nakshatra Punarvasu comes next. The name is derived from पुनर् again and वसु-the eight Vasus, the reappearance of the Vasus. The Vasus were thrown into the Ganges but have reappeared on the banks of the heavenly Ganges (Milky Way). The eight stars forming the group are κ , β , α , ι , δ , λ Geminorum and β and α Canis Minoris (Procyon). To the observer of the heavens the group distinctly resembles the figure of a horn (the hunter Mriga Vyadha or Orion's hunting horn). Punarvasu later on meant the two stars α and β Geminorum (Castor and Pollux). In still later astronomical works the star Pollux owing to its proximity to the ecliptic was designated by Punarvasu.

The nakshatra Ardra, from आर्द्र-moist, got this name not because of any meteorological significance but for

the reason that the principal star Ardra (α Orionis, Betelgeux) is situated just on the brink of the heavenly Ganges (वर्मणा)—the Via Lactea (Milky Way).

The Nakshatras Mrigasiras and Rohini have been already described. The star Rohini is Aldebaran. The car of Rohini (रोहिणी शकट) with the flag-staff somewhat bent, as if due to the speed of the car, is shown in the chart of the heavens.

The asterism Krittika got this name from its resemblance to a krittika or kartarika (from root कृत् to cut), a cutting appliance. The shape of the cutting instrument is clearly represented in the figure formed out of the stars η , γ , α , δ , ν , ϵ , ζ and \circ Persei with Pleiades forming the buttend of the sword. It was with this sword that Perseus slew the gorgon Medusa (β Persei, Algol). It was the same *parasu* or *parsu* ('परशुः पशुर्व च') with which Parasurama was equipped. The stars in Perseus also resemble a bow with which Kartikeya (from Krittika) was adorned. The star Alcyone is identified with Krittika.

Here a description of a few other stars found in ancient treatises may be useful.

Mriga Vyadha (the hunter of the Mriga) or Lubdhaka (the dog) is the star α Canis Major (Sirius.) The asterism Prajapati or Brahma is the constellation Auriga, with δ Auriga, as Prajapati and Capella (α Aurigae) the heart of Prajapati or Brahmahridaya. β Tauri is identified with the star Agni, situated in the Milky way—the celestial river, hence 'Agni living in waters' ('अग्नि' समुद्रवासनं, etc.) will be understood. In the Vedic literature we find Prajapati described as creating Mithuna ('...प्राजापतिः मिथुनम् उत्पादयते'). Brahmá is commonly associated with Agni. We have the goat for the carrier of Agni, 'ह्यगस्थः साक्षस्त्रोऽग्निः...', etc.' In all western charts Auriga (Brahmá) is depicted as carrying the kids. When the vernal equinox was happening

about this group the phenomenon of the coincidence of the vernal equinox and perihelion occurred there (B.C. 4089). Hence the constellation was named as Prajapati the originator of Mithuna (Gemini)—coincidence of the two. This is the origin of the name of the sign Mithuna.

The derivation of the nakshatra 'Bharani' has been already given. There are two triangles one formed out of the stars α, β Triangula (northern) and 41 Arietis, the other, a small one formed of the stars 35, 39, and 41 Arietis. In the second period beginning with the month of Kartika full moon in Bharani indicated the end of the year—भरणी-अवभरणी-अप-हरणी-अपहरणी...death or end of the year.

The nakshatra Aswini (from अश्व horse and suffix इन्, possessor of horses) is identified with the two prominent stars α and β Arietis, the two Aswins. The head of the two horses possessed by them are the two triangles just described. In later astronomical treatises Aswini is identified with β Arietis.

The sign Mesha (from मिष् स्वर्जयाम् to make bold) originated from the bold attitude taken by horses, also by the ram.

The asterism Revati comes next. It is derived from रव-प्रवगतिः slow motion, evidently from the observation of the movement of the vernal equinox and perihelion which were once coincident in it. By carrying our calculation back to the time when the two phenomena coincided just before the coincidence in Libra we reach the longitude of ζ Piscium (19°) from modern data, and long. 7° (according to the Hindu rate of $48''$ annual precession) which is in the portion of Revati (assuming the longitudes to be counted always from a fixed point). Revati is identified with ζ Piscium. "Ptolemy calls it δ in lustre and he is followed by Ulugh Beigh, Tycho Brahe and Hevelius. Mr. Baily says 'This star is stated in the Brit.' Catalogue to be of the 4th magnitude

but in the original observation it is nowhere stated to be more than the 5th ; and in one place it is marked as the 6th but afterwards altered to the 5th, which I have retained."—Smyth's Cycle.

Then comes the two nakshatras Purva and Uttara Bhādrapada. Bhādrapada is derived from bhadra (good) and pada (foot). It refers distinctly to the figure which is good in all respects (sides and angles all equal) a square figure (*cf.* सर्वतोभद्रमण्डल), the square in Pegasus formed of the stars β, α, γ Pegasi and α Andromedae—the first two are the Purva and the last two the Uttara Bhādrapada. The Bhādrapadas are also called in the Vedic literature by the name प्रोष्ठपद prosthā means a bedstead from प्र-उष् (warm) and स्थन् 'उषिकुषि-गार्त्तिभ्यःस्थन्।'—(one feels warmer to lie on a bedstead than on the ground)—and pada-leg—the four legs of the bedstead. The presiding deity of U. Bhādrapada is अहिर्ब्रध्नः। अहि a snake and ब्रध्न (वन्धे ब्रध्निश्च इति नक्त्) bound down—bound down (chained) by a sea monster. It refers to Reva being rescued by the Aswins, "You raised up, like soma in a ladle, Reva, who for ten nights and nine days, had lain (in a well) bound with tight bonds, wounded, immersed and suffering distress from the water." Rig Veda, I, 116-24. Compare the Greek legend of Andromeda, the Chained lady, being rescued by Perseus. From रेव (Reva) we have रेवा or रेवती (both from the root रेव्).

The name of the sign Mina (Pisces) originated from the resemblance of several stars in it to the figure of a fish (the portion cut off between two intersecting circles). The fish figure is nicely formed out of the stars $\alpha, \nu, \mu, \zeta, \epsilon, \delta, +1, \omega, \iota, \gamma$ Pisces, $\theta, \eta, \phi^2, \phi^1, \iota$ Cetus, $30, 20, \kappa$ Pisces, etc.

Now we come to the nakshatra Satabhisaj. The name is derived from शत (abundant, copious) and भेषज (water) शतं भेषजानि अस्य इति one possessing copious water. The entrance of the sun in this asterism in that ancient period

indicated the downpour of copious rain, hence the name Satabhisaj. 'सहस्र राजा वर्षाधिपतिः । नक्षत्राणां शतभिषग् वसिष्ठः । तौ देवेभ्यः कृणुतो दीर्घमायुः । शतं सहस्रा भेषजानि धत्त ॥' (Tait. Brah., 3-1-2.)

भेषज was originally a synonym of water (*vide* Yaska's Nighantu). Later on it meant medicines for the reason that medicines were then administered in their liquid extract form as being more easily assimilable by the system: and physicians administering these began to be called भिषक् ।

The hymn in the Rig Veda 1-24-9 "शतं ते राजन् भिषजः सहस्रम् उर्वी गभीरा सुमतिष्ठे अस्तु" does not mean Varuna possessing hundreds and thousands of physicians but simply having numerous water carriers (to water the earth) like the *vistis* in the towns. Varuna is derived from root वृ-आवरणे to cover. Clouds cover the sky, also darkness covers the earth at night. So Varuna is spoken of as the lord of water (जलाधिपति), also the presiding deity of night.—मित्रावरुनौ day and night. 'अहर्वेमित्रो रात्रिर्वरुणः...'—ऐतरेय ब्राह्मण—४-२-१० । The car of Usha (Dawn) goes a certain distance and stops at the banks of the river Vipasa, when with the rising of the sun the pasas (nets) of Varuna (darkness) disappear—hence the name वि-पाशा This phenomenon is beautifully sung in the Vedas. The star Satabhisaj is identified with λ Aquarius. The star Varuna is α Piscis Australis (Fomalhaut).

We come to the sign Kumbha (Aquarius). Kumbha is derived from क-जल water and उभ-पूरणे to fill. It refers to the head of the aquatic animal Makara, which is filled with water and the water again ejected out of the nostrils. It was supposed that on the entrance of the sun in this portion, the Makara ejected water stored in its head to flood the earth. Kumbha originally meant the head of the elephant (*vide* Amarakosha). The word कुम्भीर also meant a type of aquatic animal which stores and ejects

water from its head—probably the whale. In mythology we have Ganga depicted as Makarabahini (borne by the Makara). Makara is also known to the astronomers as मकर—a type of aquatic animal. ‘न तद्दिन् धनुर्नक्रश्चापि न वृश्चिको...,’ etc.—Bhaskara-Tripirasnavasana—verse 32. Nakra is an offering to Varuna “... मित्राय कुलीपयान् वरुणाय नाक्रान् ।” Yajur Veda—24-21. *Vide* discussion on Makara (ante).

The origin of the names of the signs Makara and Kumbha is now found out. Capricornus (the goat) is of later introduction. The nakshatras Sravana and Sravistha have already been dealt with. The star Sravana is identified with Altair and Sravistha or Dhanistha is the star δ Delphinus. Dhanistha is derived from धन-धान्ये having abundant food grains after the rains. When owing to the effects of precession Sravana and Sravistha were no longer the harbinger of rains, the ‘स’ was changed to ‘श’ ।

The nakshatra Abhijit has already been identified and sufficiently described.

Now comes the two Ashadas (Purva and Uttara). Ashada is derived from न (no) सह (to brook)—one that does not brook an insult—referring to the figure formed out of the stars in the group. It resembles a being with the lower portion that of a beast and the upper portion sufficiently bent back appearing like a man in a very defiant attitude ready to dart an arrow at the eagle (Aquila). In plain language it means that on the Sun's entrance into these asterisms lightning flashes previous to the approach of rains were visible. It was rightly hinted by Miss Plunket in her “Ancient Constellations and Calenders” that the hymn 27 in the 4th Mandala of the Rig Veda “when the hawk (aquila) screamed (with exultation) on his descent from heaven and (the guardians of the Soma) perceived that the Soma was (carried away)

by it, then the archer (धनुः) Krisanu pursuing with the speed of thought and stringing his bow let fly an arrow against it" refers to the archer धनुः-Sagittarius—sign Dhanus. The star δ is designated Purva and σ Sagittarius the Uttara Ashada in later astronomical works.

I shall mention here another asterism frequently referred to in ancient treatises—the nakshatra Sisumara (Gangetic porpoise) or Dhruba Matsya (the polar fish). Bhaskara mentions it in his Goladhyaya, Ch. I, sloka 10 '... भार्कन्दूनां विलोक्त्वाङ्गाध्रुवमख्य परिभ्रमम्।' It is also mentioned in the Maha Narayanopanishad 'महर्षि मण्डलानि अतिक्रम्य सूर्य सोममण्डले भित्वा . . . ध्रुवमण्डलस्य दर्शनं कृत्वा भगवन्तं ध्रुवम् अभिपूज्य ततः शिशुमारचक्रं विभिन्य शिशुमारप्रजापतिं अभ्यर्च्य' and in the Vishnu Purana 'तारामयं भगवतः शिशुमाराङ्गनि प्रभोः । दिविरापं हरिर्यत्तु तस्यपुच्छे स्थितोऽध्रुवः।'—2-9-1. Compare also Brahmanda Purana, 56-92, Vayu Purana, 52-91, Brahma Purana, 24-5.

The figure is composed of the stars β , δ , η , Draconis and γ , β , 5 and 4 Ursae Minoris and is easily recognised in the night resembling the figure of a Gangetic porpoise. Now Bhaskara in his notes on the above-quoted sloka says : 'यदा भरणीस्थो रविर्भवति तदा तस्यास्तमय-काले ध्रुवमख्य तिर्यकस्थो भवति । तस्यमुखतारापश्चिमतः । पुच्छतारापूर्वतः । तदा मुख तारा-सूत्रे रविः इत्यर्थः । अथ निशावसाने मुखतारा परिवर्त्य पूर्वतो याति । पुच्छतारा पश्चिमतो याति । ततोमुखतारा-सूत्रगतस्य एव अक्षस्य उदयो दृश्यते । अतो द्वौ द्वौ सूर्यौ इति न उपपद्यते ।' When the sun enters Bharani at the time of his setting, the polar fish is seen in a slanting position. The star in its mouth is towards the west and that in the tail is towards the east. At that time the sun is in the line drawn through its mouth and the pole. Again in the morning the star in the mouth goes towards the east and the star in the tail is towards the west. Therefore the rising of the sun is seen in the line drawn through the mouth (of the polar fish). Therefore it is proved there are no two suns, etc.

From this it is evident that the pole star passed through the mouth of the figure. The star δ Ursae Minoris (a variable one) in the mouth of the fish was the pole star about B. C. 200. But from the sloka in the Vishnu Purana we learn that the pole lies towards the tail of the **Sisumara**. The star in the tail is β Draconis. We have Abhijit (Vega) at a short distance from the tail and it was once the pole star. It is evident the Vishnu Purana describes a very ancient position, as its very name implies (Purana—puratana—ancient), when the pole was in the tail of the Sisumara (the Gangetic porpoise)—the polar fish. The Brahmanda Purana on the other hand states the latest position when the pole was in the mouth of the Sisumara. 'उत्तानपादस्तस्याथ विज्ञेयो ह्युत्तरो हनुः।' Uttanapada (β Ursae Minoris)—father of Dhruva, is the upper jaw of the Sisumara. The stars β , ξ , γ Draconis and ι Hercules form another small fish just touching the bigger one, of which the star ι Hercules was the pole star after Vega, though not so remarkable as Abhijit—being a fourth magnitude star.

The Sisumara is also referred to in the Rig Veda as having been yoked with the Vrishabha (*Taurus*)—an apparent incongruity, the Sisumara (porpoise) being an aquatic animal and the bull a land one. 'To the car were yoked together the bull (Vrishabha) and the Sisumara to revolve round.' Rig., I-116-18.—a clear reference to the sign Vrishā (Mriga—Orion). This happened when the pole was near about ι Hercules in B. C. 9000 about.

The Saptarshis (the Seven Rishis) are the stars α , β , γ , δ , ϵ , ζ , η Ursae Majoris (the Great Bear).

Now it is worth while entering into the discussion of the Revolution of the Rishis as distinctly noticed in ancient treatises. A short extract from Colebrooke's Essay on the Equinoxes will introduce the subject.

“ ... Varaha Mihira has a chapter in the Varahi Samhita expressly on the subject of the supposed motion of the Rishis. He begins by announcing his intention of stating their revolution conformably with the doctrine of Vriddha Garga, and proceeds as follows. ‘When king Yudhisthira ruled the earth the ‘munis’ were in Magha’ and the period of the era of that king is 2526 years. They remain for a hundred years in each asterism being connected with the particular nakshatra to which when it rises in the east the line of their rising is directed’ :—‘आसन् मघासु मुनयः शासति पृथ्वीम् युधिष्ठिरे नृपतौ । षड्विंशतिवर्षाद्युतः शकालस्तस्य राज्यस्य । एकैकस्मिन् ऋते शतं शतं ते चरन्ति वर्षाणाम् ।...’

The commentator, Bhattotpala, supports the text of his author by quotations from Vriddha Garga and Kasyapa. ‘At the junction of Kali and Dwapara ages’ says Garga, ‘the virtuous sages ... stood at the asterism over which the Pitris preside. That is at Magha.’ ‘The mighty sages,’ says Kasyapa ‘abide during a hundred years in each asterism,’

The revolution of the seven Rishis, and its periods, are noticed in the Puranas. The following passage is from the Sri Bhagabata, Book XII, Ch. 2.....of the seven Rishis, two are first perceived, rising in the sky; and the asterism, which is observed to be at night even with the middle of these two stars, is that with which the Rishis are united, and they remain so during a hundred years of men. In your (Parikshit’s) time, and at this moment, they are situated in Magha. ...When the seven Rishis were in Magha the Kali age ... began.

A similar passage is found in the Vishnu Purana, and a similar exposition of it is given by the commentator Ratnagarbha. The Matsya Purana contains a passage to the like effect; In the Brahma Siddhanta of Sakalyathe supposed motion of the seven Rishis is thus

observed. 'At the commencement of the Yuga, Kratu was near the star sacred to Vishnu at the beginning of the asterism.... Their motion is eight liptas (8') in a year (should be 0'8').... For moving in the north into different positions the sages employ 2700 years in revolving through the assemblage of asterisms.... Muniswara in his commentary on the Siromoni cites some of the passages quoted above, and remarks, that Bhaskara has omitted this topic on account of contradictory opinions concerning it,

The same author in his own compilation entitled Siddhanta Sarvabhauma has entered more fully into this subject. He observed that the Seven Rishis are not, like other stars, attached by spikes to the solid ring of the ecliptic, but revolve in a small circle round the northern pole of the ecliptic, moving by their own power in the ethereal sphere above Saturn, but below the sphere of the stars.... Muniswara supports this mode of calculation on the authority of Sakalya, against Varaha Mihira and Lalla, and affirms that it agrees with the phenomenon as observable at the period of his compilation.

Kamalakara notices the opinion delivered in the Siddhanta Sarvabhauma; but observes that no such motion of the stars is perceptible. Remarking, however, that the authority of the Puranas and Samhitas which affirm their revolution is incontrovertible, he reconciles faith and experience by saying that the stars themselves are fixed, but the Seven Rishis are invisible deities, who perform the stated revolutions in the period specified.

If Kamalakara's notion be adopted, no difficulty remains, yet it can hardly be supposed, that Varaha Mihira and Lalla intended to describe revolutions of invisible beings. If then it be allowed, that they have attributed to the stars themselves an imaginary revolution grounded on an erroneous theory, a probable inference may be thence drawn as to the period when those authors lived,

provided one position be conceded ; namely, that the rules stated by them, gave a result not grossly wrong at the respective periods when they wrote. Indeed it can scarcely be supposed, that authors, who, like the celebrated astronomers in question, were not mere transcribers and compilers, should have exhibited rules of computation, which did not approach to the truth, at the period when they were proposed."

'In the preceding passages with respect to the Rishis quoted by Colebrooke from various astronomical works of the Hindus, the writers agree in the common mistake of the supposed motion of the line of Rishis, and in the opinion that a solstice moves through each asterism in 100 years ; but we can only regard these mutilated fragments of a nearly perfect theory as having had a common origin in a remote age. We may suppose that they have been handed down from the same Jyotisha family by its scattered descendants and that the original doctrines have lost their true form, from repeated transcripts, during long periods of time, and this liability to error would be increased by the complex nature of the subject without sufficient explanation. In short the rate of motion of the solstices originally known and so near the truth, became lost to the successors of the earliest astronomers.'—Brennand, *Hindu Astronomy*, p. 76.

In fact it was not the Seven Rishis alone that were moving but all the Rikshas (nakshatras—stars) were moving with respect to the equinoxes or the solstices. The seven prominent stars were also termed the 'Rikshas'—the bear from which we have the Great Bear of the westerners. The ancients really meant to say that all the Rikshas move with respect to the equinoxes or the solstices, or, which comes to the same thing—they move away with respect to the seasons. But Riksha also meaning a bear—the motion was mistakenly applied to the Great Bear (Saptarshis) alone.

“The early astronomers, however, considered the equinoxes to be immovable; and ascribed the change of distance of the stars from it, to a real motion of the orb of the fixed stars which they supposed to have a slow revolution about the poles of the ecliptic in the Platonic period of 25920 years, a space not remotely different from that produced by moderns, from other principles.”—Smyth’s Cycle, Vol. II, p. 227. Of course owing to the proper motion of the stars in the group, the relative positions of the same may sensibly change, but that does not mean that they alone will revolve round the pole of the ecliptic in 2700 years. “Of the seven brightest stars of the Plough (Great Bear—Saptarshis), five have proper motions in the same direction and of nearly the same amount while α and γ have a proper motion in the opposite direction. These movements though slight, are sufficient sensibly to alter to the eye the relative positions of these stars after the lapse of several centuries.”—Heath, Atlas of Popular Astronomy, p. 110.

During the Vedic age the word ‘Rishi’ also meant the seasons. “...The six seasons are also termed Rishis, rishayah, which Sayana explains by ‘gantarah’-goers: in what sense does not appear.”—Wilson, Translation of the Rig Veda, Vol. II, p. 132 (note).

As is well known to all, the seasons come and go away. Hence ‘Ritus’ were termed Rishis from ऋष गतौ (moving). In a solar year there are twelve lunar months and some days more. Counting from the vernal equinox two lunar months make one season. ‘सूर्यात् मासान् षडभ्यस्तान् विद्यात् चान्द्रमसान् ऋतून् ।’—Vedanga Jyotisha. Therefore in a year there are six seasons plus some days more. Hence the seasons are termed seven of which six are in pairs and the seventh solitary.

‘सार्कं जानां समयम् आहुः एकजं षड् इदं यमा ऋषयो देवजा इति ।’
(Rig Veda, I-164-15.)

‘Of those that are born together, the sages have called the seventh single born, for six are twins and born of heaven (gods).’

‘समास्तुवाग्म ऋतवो वर्धयन्तु संवत्सरा ऋषयो यानि सत्या संदिध्येन दीदिहि रोचनेन बिष्वा आभाहि प्रदिशस्वतस्तः ॥’—Atharva Veda, 2-C. ‘O Agni, let the summer season increase thee; the years, seasons, what things are true (increase thee). Shine thou with the heavenly bright space, illuminate all the four directions.’

The ancients began to observe that this or that season starts on the sun’s entrance into this or that asterism. Later on they began to observe the same season to occur on the sun’s entrance into a different asterism. In this way they concluded that the Rishis (seasons) are moving. The first effect of precession observed is the falling back of the seasons with respect to the position of the sun among the asterisms. This they carefully noted and formulated that the seasons fall back by one nakshatra in one thousand years, *i.e.*, twenty-seven thousand years for one complete revolution. The movement of the Rikshas (the stars) starting from an equinox or solstice and return to the same position—or what comes to the same thing as the movement of the Rishis (the seasons) was confused later on with the movement of the ‘Rikshas’—the bear—the Great Bear—Ursa Major—the Seven Rishis (the word Rishi is pronounced in upper India as *rikhi*) alone, the original sense having been lost.

“...From the extract above given it will be seen that the several writers refer to a motion which they themselves evidently did not understand, but which they were endeavouring to explain from traditional doctrine received from previous astronomers, to whom the subject was really clear....It will be observed that the astronomers of the period between the 16th and 14th centuries before the Christian era had made many discoveries and amongst

others this that the solstitial colure was moving backwards along the signs....Now what is more natural than that omissions or mistakes should be made in the numerous copies of the statements of the original astronomers, who lived more than 28 centuries ago, or that a cipher should have been lost, or even a dot (which we are told ancient writers used in lieu of a cipher) at the end of the number, and that the modern Hindu writers should have been misled in stating 100 instead of 1000 years, 2700 years for a revolution instead of 27000? ”—Brennand, *Hindu Astronomy*, pp. 73-76.

That this really happened will be evident on looking at the number of revolutions of the sun's apsis in a kalpa as given in the ancient treatises. In place of 38700 it is stated as 387 only, thus making the annual movement of the apsis 0.1161" in place of 11.61". Even now-a-days such mistakes are not uncommon. As an example one will find in the *General Elements of the Earth* in Smyth's Cycle, Vol. I, a book printed with great care, the complete tropical revolution of the apsides stated to occur in 20984 years in place of 20984 years.

The word Rishi in its original sense we have still preserved among the westerners when they styled 'recession of equinoxes' to denote 'precession of equinoxes' (*vide* Proctor's *Old and New Astronomy*) (Recession from Lat. recessus—to move). Now-a-days 'precession of equinoxes' is more generally used. This movement of the Rishis is nothing but the 'Recession of the equinoxes or the solstices.' Aryabhatta mentions the number of revolutions of the Saptarshis in a Kalpa to be 1599998 (this is the same as that of Parasara quoted by Aryabhatta). Thus the number of years for one revolution comes out to be slightly more than 2700. The correct figure should be 27000 years. Aryabhatta (also Parasara quoted by him) mentions the period of attaining maximum

libration (Ayana bhagana) of 24 or 27 degrees immediately after. The number of librations in a kalpa according to Aryabhatta is 578159 and that according to Parasara 581709, slightly different from that in the Surya Siddhanta which is 600000. The adoption of the Saptarshi (tropical or mean solar) year or the Nirayana (apparent solar) year by the Hindus requires the solar intercalary month to be taken account of. In the Vedic literature we have three types of intercalary month mentioned—संसर्प, मलिन्नुच and अंहस्यति. Samsarpa and Malimlucha have been rightly identified with the two types of lunar intercalary months, Kshaya (क्षय) and Malamasa. Amhaspati is Ahaspati, Aharpati or Divapati (Sun)—the solar intercalary month; compare—‘संसर्पाय स्वाहा चन्द्राय स्वाहा ज्योतिषे स्वाहा मलिन्नुचाय स्वाहा दिवापतये स्वाहा ।’—वाजसनेय संहिता, २२-३०, and the very next mantra—‘मधवे स्वाहा माधवाय स्वाहा शुक्राय स्वाहा शुचये स्वाहा.....अंहस्यतये स्वाहा ।’ The derivation of अंहस्यति is similar to that of शिशुमार । Both are Vedic forms for अंहस्यति and शिशुमार (शिशून् मारयति इति injuring children). *Vide* also Rig Veda (1-25-8) on the knowledge of the intercalary month possessed by the ancients. ‘... We cannot doubt the correctness of the Scholiast’s conclusion, that the thirteenth, the supplementary, or intercalary month of the Hindu luni-solar year is alluded to;.... The passage is important as indicating the concurrent use of the lunar and the solar years at this period (Vedic), and the method of adjusting the one to the other.’—Wilson, Translation of the Rig Veda, 1st Vol., pp. 65-66, foot-note. The length of the apparent solar year is given by Srisena in his Romaka Siddhanta and is quoted by Varaha in his Panchasiddhantika. Srisena took the length of the solar year from Iata (acharya)—an ancient Vedic astronomer. *Vide* Introduction to Panchasiddhantika by Thibaut and Dvivedi, p. xxvi. Thus the Recessional period (Saptarshi Bhagana)

and the period of Libration (Ayana Bhagana) were distinct with the Hindu astronomers. Bhaskara and others mixed up the two and understood the degrees of Libration to be the same as the degrees of Precession.

“The doctrine in question (the Libration of Equinoxes) found advocates formerly among the astronomers of Europe and Arabia. Arzachel, a Spaniard, and a mathematician of the 11th century, author of a treatise entitled ‘The observation of the Obliquity of the Zodiac,’ affirmed a libration or trepidation in longitude within the limits of 10° East and West at the rate of a degree in seventy-five years. Two centuries after him, Thabit Ben Korah, an astronomer, assigned to this supposed trepidation the limits of 22° East and West.

The notion of a trepidation in longitude, but *at a rate not equable*,¹ had been entertained by the astronomers who compiled the Alphonsine Tables, though Alphonsus himself was subsequently led to the adoption of a correcter opinion and to the consequent alteration of the tables first published by him”—Colebrooke, *Essay on the Equinoxes*, p. 337.

The theory of the Libration of Equinoxes was prevalent in Europe till the time of Tycho Brahe (1600 A. D.) who “as an incidental though valuable result, demonstrated the unreality of the supposed trepidation of the equinoxes”—*Encyclopædia Britannica*, Tycho Brahe, 10th edition, p. 811.

It is evident from all available information that this theory of the Libration of Equinoxes was world-wide

¹ Ranganath speaks of this motion as *bilakshana*—possessed of peculiar characteristics, but being unable to explain the phenomenon ascribed it to the will of God ‘ईश्वरेच्छया’.

till the 16th century, when it was rejected in Europe by Tycho Brahe, followed by others who supported him, in not being able to explain the phenomenon.¹

Now we come to the question of the Rishis (Solstice) being in Magha during the reign of Yudhisthira. The Kali era begins from B. C. 3102, 17th February. This is exactly the year of the Deluge (3102 B. C. February 18). (*Vide* Herschel's Outlines of Astronomy, p. 679, in the table 'Intervals in days between the commencement of the Julian period and that of some other remarkable chronological and astronomical eras.') Now from 3102 B. C. to 1923 A. D. we have 5024 years. During this time vernal equinox receded ($5024 \times 48''$) or 67 degrees. Therefore summer solstice occurred at $(90 + 67)$ or 157 degrees. Starting from ζ Piscium (Revati) and allowing $13\frac{1}{3}$ degrees for the portion of each nakshatra, the end of Magha the tenth nakshatra is $133\frac{1}{3}$ degrees distant. But ζ Piscium being 19 degrees away from the present vernal equinox the distance of the end of Magha from the present vernal equinox is $(133\frac{1}{3} + 19)$ or $152\frac{1}{3}$ degrees. Therefore actual summer solstice in 3102 B. C. was situated $(157 - 152\frac{1}{3}) 4\frac{2}{3}$ degrees after the end of Magha. But as is well known if the correction for refraction is not made the vernal equinox (day and night equal) happens about three days

¹ A simple explanation of the theory of the Libration of Equinoxes may not be out of place here. The apparent sun moves along the ecliptic. The fictitious mean sun is, as usual, supposed to move along the equator. There are four days in a year on which the R. A. of the apparent sun equals the longitude of the mean sun, *i.e.*, the two suns are on the same hour circle. If any of the four points on the ecliptic, where on the arrival of the apparent sun the above phenomenon happens, be taken as the initial point, the solstitial or the equinoctial point will be found to librate about 24 or 27 degrees, about this initial point and *vice versa*. For further details *vide* 'Libration of Equinoxes,' Notes on Indian Astronomy, Journal of the Department of Letters, Vol. V.

earlier. Therefore the summer solstice might well be declared to have occurred at longitude (157—3) or 154°, and this is only (154-152 $\frac{1}{3}$) 1 $\frac{2}{3}$ degrees after Magha. For convenience it is assumed that the longitudes are always counted from the same fixed point. “Had a fixed star been chosen as the origin of longitudes, they would have been invariable.”—Herschel, *Outlines of Astronomy*, p. 203. For a difference of 1 $\frac{2}{3}$ degrees we need not bother, especially when we remember that even with some old European observations the rate of precession was taken to be 45". “The Rev. Charles Turnor (Newtonian Turnor) has recently lent me a very valuable MS. Almanac of 1310,.....Among its contents is a list of 35 principal stars.....and their constellation places, brought up from Ptolemy’s catalogue by applying 45" for the annual change of longitude.....”—Smyth’s *Cycle*, Vol. II, p. 525. Bentley also took 48·2" for the mean annual precession allowing 46·2" and 50·2" to be the minimum and maximum values. In Smyth’s *Cycle* published in 1811 it is written that “*arsae Minoris* (Polaris) will be the pole star in 2095 A. D. whereas in Schurig’s *Star Atlas* recently published we find the year stated as 2115 A. D., a difference of 20 years in a prediction for a period of about 250 years.

There is an important passage in the *Satapatha Brahmana* which reveals the time of the compilation of the same, and incidentally the time of Yudhisthira. “*कृत्तिकासु अग्नी आदधोत ।... एताह वै प्राचैदिशो न चवन्ते । सर्वानि ह वा अन्यानि नक्षत्राणि प्राचैःदिशः चवन्ते । तत् प्राच्याम् एवास्य एतद्दिश्याद्विती भवतस्तस्मात् कृत्तिकासु आदधोत ।*”...शतपथ ब्राह्मण २-१-२ “.... These the *Krittikas* do not deviate from due east. All other *nakshatras* do deviate from the east, etc.” Now the present longitude of *Alcyone* (*Krittika*) is 59°. The latitude of the same is 4° N. *Alcyone* or for the matter of that any star to be due east must be on the

equator. This happened when Alcyone was at least 9° west of the vernal equinoctial point. As the obliquity is somewhat variable, moreover the group Pleiades was meant by the Rishis, so I have here stated the minimum value. Therefore the vernal equinoctial point has by this time moved ($59^\circ + 9^\circ$) or 68° to the west. The impossibility of the vernal equinoctial point librating 27° about ζ Piscium (Revati) is apparent. Now assuming 72 years for the precessional movement of 1° , (68×72) or 4896 years have elapsed since the book was compiled. Hence the date of the compilation of the Satapatha Brahmana is about B.C. 3000. In this book we find the name of King Janamejaya, great-grandson of Yudhis-thira. 'एतेन ह इन्द्रोतो देवायः शौनकः जनमेजयं पारिक्षितं याजयाञ्चकार।' शतपथ—१३-५-४। Therefore King Yudhis-thira might well have lived (5024—4896) or 128 years prior (that is in 3102 B.C.) to the date of the Satapatha Brahmana. "In the Ayeen-i-Akbari, II, p. 110, it is stated that the great war 'happened in the end of the Dwapara Yuga, 105 years prior to the commencement of the Kali Yuga, being 4831 years anterior to the fortieth year of the present reign (that of Akbar).' The fortieth year of Akbar was 1595 A. D. Therefore the great war happened in (4831—1595) or 3236 B. C. Therefore the commencement of the Kali Yuga was (3236—105) or 3131 B. C. (Abul Fazl)."—Brennand, Hindu Astronomy p. 118

In the Vedanga Jyotisha we find another position of the colures described. The summer solstice was then situated in the middle of Aslesha, *i.e.*, at longitude $132\frac{1}{2}^\circ$. Allowing the common difference of about 3 degrees, actual solstice was then in long. $135\frac{1}{2}^\circ$ about. This happened about B. C. 1478 (taking the annual precession to be $48''$), the time of the last edition (revision) of the Vedanga Jyotisha. (*Vide* pp. 54-55.)

Prof. Max Muller's apprehensions, 'I feel bound to remark that unless there were internal evidences that the Vedic hymns reached back to that remote antiquity this passage of the Jyotisha would by itself carry no weight whatsoever,' are groundless in view of the fact that the learned professor cannot expect an astronomical treatise extant from ten-thousand years before, to be still showing the position the colures occupied when the treatise was first composed. The subject having to be taught from actual observations it was natural to have changes introduced from time to time by the revisers. 'The existence of the different versions of the Vedanga Jyotisha and the obscurity into which some of its verses are still shrouded render it rather difficult to meet the above objection, especially as it is a side attack on the antiquity of the work with an admission that the position of the colures might have been recorded in the work from real tradition current in the time of the author.'—Tilak, Orion, p. 39.

Coming to Varaha Mihira's time we find Varaha observing summer solstice entering Punarvasu.¹ From the end of Punarvasu to the end of Magha we have three nakshatras (a distance of 40 degrees). The solstitial colure to fall back by three nakshatras would require 3000 years. Varaha Mihira states that Saka Era started 2526 years after the era of Yudhisthira and he was writing about 500 Saka year. So that the number of years elapsed

1

आग्नेषार्धादासीद्यदा निरतिः किलोत्थकिरणस्य ।
युक्तमयनं तदासीत् साम्प्रतमयनं पुनर्वसुतः ॥

also

आग्नेषार्धाद्विषुसुत्तरमयनं रवेर्धनिष्ठाद्यम् ।
नूनं कदाचिदासीद्येनोक्तं पूर्वमाग्नेष ॥
साम्प्रतमयनं सवितुः कर्कटाद्यं समादितस्याव्यत ।
उक्ताभागे विहतिः प्रत्यक्षपरीक्षणेऽप्येति ॥

2526+500), about 3000 years, accounts for the solstice to enter Punarvasu. Counting as usual, the end of Punarvasu is $112\frac{1}{3}$ deg. from the present vernal equinox. Therefore apparent vernal equinox was happening then at $(112\frac{1}{3}-90)$ or $22\frac{1}{3}$ deg. or vernal equinox as understood by moderners was at longitude $25\frac{1}{3}$ degrees.¹ Even allowing precession² at 50" annually, 1824 years have elapsed since then. That is the present Saka year should be at least (1824+500) or 2324. 5024 Kali era corresponds to (5024-2526) or 2498 Saka ; whereas the present Saka year is stated to be 1844. An error of 500 years in the counting of the Saka era is incredible. The fact is, 'Saka kála' of Varaha is nothing but the 'Sákya kála,' the era of Sákya Muni or Buddha. For this suggestion I am indebted to Prof. Bhandarkar of the Calcutta University. It gets support from the following: "And the ultimate basis of them is to be found in my opinion in the point that in early times, alongside of the words Saka, Śaka, as a tribal name, there were in use the forms Saka, Śaka=Sakka, Śakka, corruptions of Sakya, 'a Buddhist.—Dr. Fleet, 'The Date of Kanishka,' Journal of the Royal Asiatic Society, 1913, p. 994. The date of Buddha's Nirvana is 544 B. C. Therefore the present Buddha year is 2466, a difference of 32 years only, from Varaha's Saka kala. Compare Brenand, Hindu Astronomy, p. 118, footnote. There 605 B. C. refers to the beginning of the Saka (Sakya) era and not the age of Yudhisthira and Parasara as mistaken by Brënnand. The mistake was here 'the period of

¹ That vernal equinox was happening in longitude $25\frac{1}{3}^{\circ}$, will be evident on looking at the position of some of the stars given by Varaha. Regulus (Magha) and Spica (Chitra) being two first magnitude stars could be observed after sunset on the apparent vernal equinoctial day. The longitude of Magha is given as 126° . The present longitude of Magha being $148^{\circ}45'$ (about), the apparent vernal equinox was then in longitude $(148^{\circ}45'-126^{\circ})$ or $22^{\circ}45'$. Therefore the true vernal equinox of moderners was then at about long. $25^{\circ}45'$.

Yudhisthira is here stated to be 2526....” Whereas what Varaha meant was that 2526 years had elapsed from the time of Yudhisthira to the beginning of the Saka kala. Kalidasa the poet wrote his ‘Jyotirvidabharana’ in 3068 Kali era, *i.e.*, in (3102—3068) or 34 B. C. Vikramaditya ruling about 58 B. C., Kalidasa might well have adorned his court. Kalidasa also mentions the date 445 Saka in this book which refers to the Sakya era then prevalent—this is Varaha’s ‘Saka kala.’ 445 Saka may be the date of Kalidasa’s birth. Buddha era (the Nirvana era) starting from 544 B.C., Kalidasa was born in (544—445) or 99 B. C. Varaha Mihira mentions the date 427 Saka, which is perhaps the year of his birth. In that case he was born in (544—427) or 117 B. C. The Buddhist influence during Vikrama’s reign will be apparent from the name of क्षपणका (क्षपयति विषयरागं इति—a term applicable only to Buddhist monks and nuns)—क्षणा or खना who adorned his court. That Varaha Mihira was one of the nine gems adorning the Court of King Vikramaditya who started the Samvat era (58 B. C.), will now be evident to all.¹ The present Saka era was started by a king who was not a Saka. “In one sense I agree with Dr. Thomas in regarding this as being only nominally a Saka era. I hold that the founder of it was not a Saka, and that the name of the Sakas became attached to it under the influence of a mistaken tradition. But that is not his meaning. He has stated a ‘rather surprising fact,’ which certainly does startle me; namely that this

¹ Bhaskara wrote his treatise in 1072 Saka. The degrees of precession—the distance of the vernal equinox from ζ Piscium—given by him is stated to be about 11°. Bhaskara living about 800 years before, actual vernal equinox was then at long. 11° or 8° behind ζ Piscium (the present longitude of ζ Piscium being 19°). But the apparent vernal equinox happening about 3 days earlier, Bhaskara got the longitude of ζ piscium to be about (8+3) or 11°. Evidently Bhaskara used the current Saka era.

era was 'known in Southern India (in other provinces also) by the name of the people the Sakas, whose overthrow it commemorated.' He has quoted Alberuni as his authority for this statement. And he has taken it to be literally true—(if it should prove to be the case that Kanishka began to reign in A. D. 78)—that an overthrow of the Sakas is the event from which the era dates."—Dr. Fleet, 'The Date of Kanishka,' Journal of the Royal Asiatic Society, 1913, pp. 988-989.

Here a short description of the ancient idea of time will be useful. The words denoting time in Hindu astronomy are 'घटी' (Ghati), नाड़ी (Nadi) and दण्ड (Danda). नाड़ी (Nadi) is derived from नल (tube). The respirations through the nasal tube as well as the pulsation in the radial artery (tube), suggesting the idea. Six Pranas make one Pala or Vinadi ((वनाड़ी) (pala' from पल गती—movement; compare 'pulse' from 'pello'—beating). Sixty Palas make one Nadi (नाड़ी) or Ghati (घटी). Sixty of these Nadis or Ghatis make one day (24 hours). So that one Nadi is equivalent to 360 Pranas. But one Nadi being one-sixtieth of a day is equal to 24 minutes. Therefore 24 minutes being equal to 360 Pranas, one minute equals 15 Pranas. The ancients perceived that there were 15 respirations per minute in a man in perfect health. So that one Prana is equivalent to one respiration or four seconds in time. This makes the total respirations in a day equal to $(6 \times 60 \times 60)$ or 21600, a number well known to the Hindus.¹ The ancients also

¹ अथाह अन्धस परि-वर्त्तमानसिद्धदासीनो यदि वा रूपमपि ।

अहोरात्राभ्यां पुरुषः समैन कतिकृत्वः प्राणिति चाप चानिति इति ।

तदेव श्लोकः प्रत्युक्तः—

शतं शतानि पुरुषः सामनाष्टीशता यन्मृतं तददन्ति ।

अहोरात्राभ्यां पुरुषः समैन तावत् कृत्वः प्राणिति चाप चानिति इति ।

perceived that one respiration is equivalent to 5 pulsations. That is in a minute there are (15×5) or 75 pulsations. So that in a day there are (5×21600) or 108000 pulsations. The number 108 is well known to the Hindus from their religious point of view. It is here worth remembering that the period of the sidereal revolution of the apsis is 108000 years, one-fifth of this or 21600 years is the period of the tropical revolution of the same, one-fourth of the same period or 27000 years is the precessional period (Saptarshi Bhagana). The word 'ghati' arose from घट a vessel also from घट-संचालिते striking, from which we have घड़ो a bell that is struck at regular intervals, later on, the clocks or watches, for their striking hours or tickling at intervals. So that ancient astronomers in sound health measured small intervals of time by counting their respirations or for more accuracy, their pulsations. Galileo discovered the isochronism of the pendulum by timing the swings of a candelabrum with his own pulse beats. Nadi and Ghati consequently indicate mean time. The word Danda (दण्ड) indicates a rod—gnomon. Time measured from this will be apparent solar time (this may be done during certain hours of the day only, when the sky is clear). The significance of these terms were well known to the ancient astronomers who had to deal with these from the practical point of view. At present we mix up all these terms, as if they denote the same thing.

The ancients determined the equinox with the help of a time keeper that was filled in exactly twelve hours. The Mayura (peacock) Jantra is mentioned in the Suryya Siddhanta. The cock emptied from its mouth one earthen ball every 24 minutes till 60 of these were emptied in a day. The description is something like that of a water clock which was presented by Harun-al-Raschid to Charlemagne. "Kaliph Harun-al-Raschid was the richest and most potent sovereign of his

time and reigned from 785 to 808 A. D. He sent a splendid embassy to Charlemagne, his ally, which among other presents brought a water clock, in the dial of which were twelve small doors, forming the divisions of hours. Each of these doors opened at the time it marked, and let out little balls which falling on a bell of brass, struck the hours. The doors continued open till twelve o'clock, when twelve knights, mounted on horseback, came out together, paraded round the dial, shut all the doors and returned to their apartments.'—Smyth's Cycle, Vol. I, p. 26. In the determinations with the help of a gnomon also, we shall allow a margin of about two days. "The exact number of days elapsing between the beginning of one solar year as determined in the manner above shown by the winter solstice and the beginning of the next solar year, might perhaps be determined within a day or two, but not much more accurately.....Continuing their observations and keeping their records for many hundred years in succession, the ancients had opportunities of correcting their first rough estimations to a degree which would be marvellous were it the work of but a single generation."—Proctor, 'Old and New Astronomy,' p. 119. With these practical difficulties we shall always allow a margin of about two or three degrees with all ancient determinations.¹

The movement of the Rishis in 27000 years was so well known at one time that an era was started by name 'Saptarshi Samvat,' but perhaps during the dark age through which India passed, errors were introduced.

'The Saptarshi reckoning is used in Kashmir and in the Kangra district and some of the Hill states on the south-east of Kashmir; some nine centuries ago it was

¹ The stars having to be observed on clear dark nights, some about the equinoxes and some about the solstices introduce much difficulty.

also in use in the Punjab, and apparently in Sind. In addition to being cited as Saptarshi Samvatsara "the year so and so of the Saptarshis" and Sastra Samvatsara (the year so and so of the Scriptures) it is found mentioned as Lqkakala (the time or era of the people)...The years of this reckoning are lunar Chaitradi and the months are Purnimanta (ending with the full moon). As matters stand now the reckoning has a theoretical initial point in 3077 B. C. The idea at the bottom of this reckoning is a belief that the Saptarshis, the seven Rishis, or Saints Marichi and others, were translated into heaven and became the stars of the constellation Ursa Major in 3076 (or 3077 B. C.) and that these stars possess an independent movement of their own which referred to the ecliptic carries them round at the rate of hundred years for each nakshatra or twenty-seventh division of the circle. Theoretically therefore the Saptarshi reckoning consists of cycles of 2700 years ; and the numbering of the years should run from 1 to 2700 and begin afresh The almanacs show both the figures of the century and the full figures of the entire reckoning which is treated as running from 3076 B. C. not from 376 B. C. as the commencement of a new cycle the second" —Enc. Brit. (10th Ed.), Hindu Chronology, p. 499. If it is found on enquiry that the error really started from 3077 B. C. it is possible that after the Great Deluge (3102 B. C.) there was so much havoc wrought on a great portion of the earth that it was only with difficulty that the Sacred Scriptures (the *edas*) could be preserved, other manuscripts being either spoiled, soiled or lost. It took a long time for the remnants to settle and by that time these truths were partially lost. As an example of what even a small flood can do we have the recent one in North Bengal for a testimony. The other possibility of the error having been introduced is, during the dark age

through which India passed. "With the death of Bhaskara, the living breath of Mathematical science parted from India. Repeated invasions by a barbarous nation poured forth an abundance of calamity and in the troubled times the sacred muse of learning fled, and hid herself among a herd of commentators in the Deccan. In the dark age that followed amidst the petty dissensions of a host of semi-independent states, we find intellectual effort at a standstill;".....Jogesh Chandra Roy's Introduction to Siddhanta Darpana by Chandra Sekhara, p. 13. The Greek invasion contributed to the downfall in a great measure. "We are told that Ptolemy Soter, the favourite general of Alexander, was a great lover of literature and science. He had a passion for the collection of manuscripts, and had ample opportunities for the indulgence of this favourite pursuit, in the campaign in Asia, the literary wealth of which he acquired, and the manuscripts of which he collected as the spoil of war, and carried away from its palaces and temples.

* * * * *

It is said of Uergetes that he adopted a most unscrupulous method, of adding to the (Alexandrian) library, as for instance, that he seized books imported into Egypt from neighbouring countries and having caused them to be copied, returned the copies to the owners keeping the originals for the library.

* * * * *

The aid which must have been afforded by the library to the philosophers of the Alexandrian School (Euclid, Archimedes, Hipparchus, Ptolemy, etc.) is incalculable. To state the degree in which the more ancient sciences were embodied in the writings of this period is impossible. The subject has in a great measure, been avoided, and modern writers have been content to ascribe to eminent

men of the Alexandrian School discoveries which were in fact made long before that school was established.

It cannot be doubted that, stored in the library, were many manuscripts, containing the wisdom of ancient Eastern nations, of which there is now no record, but which must have been translated and embodied in the works of Greek authors at about this period."—Brennand, *Hindu Astronomy*, pp. 95-97.

That the Saptarshi reckoning was really meant to continue to one thousand years for one Nakshatra will be evident from the following consideration.

"Dr. Buchanan in his journey to Malayala (Sept. 1800) states that the astronomers there reckoned by cycles of 1000 years from Parasurama and of the then current cycle 976 years were expired in Sept. 1800 and that 2976 must have elapsed from the epoch of Parasurama to the year 800 A. D. from which it is concluded that the epoch of that prince is 1176 B. C."—Brennand, *Hindu Astronomy*. Without going into the detail whether this 'Parasurama Chakra' or 'Sahasra Samvatsara' as it is called (*vide* Viswakosha by Nagendranath Vasu) really started from 1176 B. C. or several years before (since it is well known Parasurama being prior to Yudhisthira cannot reign in 1176 B. C.), it is worth remembering that an era started during his auspices cannot go at variance with the 'Saptarshi' or 'Sastra Samvatsara' which was also evidently intended to run to 1000 years for one nakshatra.

In this connection another point requires careful attention. During the Vedic period some 'Yajnas' were directed to be continued up to one thousand years :—

'विश्वसृजाम अयनं सहस्रसंवत्सरं इति'—आश्वलायन श्रौतसूत्र उक्तं षट्क-६, also in other books. Now it is known to us that the yajnas were the symbolical performances of the annual course of the sun. 'सम्बत्सरो वै प्रजापतिः । प्रजापति र्यज्ञः ।' etc. Therefore this continuation of the yajna till one thousand

years had the evident meaning of the equinoxes or the solstices remaining in one nakshatra during that period. When one nakshatra was passed over, one yajna was completed and a fresh one started. 'देवा इ वै सहस्र संवत्सराय दिदौक्षिरे।'...शतपथब्राह्मण—१२.३.३. After a long period this significance of the sacrifices was lost. Coming to the later period we find the Mimansakas discussing the possibility of performing a sacrifice till one thousand years. 'यथा सहस्रवार्षिकसत्रस्य असम्भवात्मा आलक्ष्य मौमांसायां अधिकारणम् आरभ्य सिद्धान्तितम्!'—Satyabrata Samasrami in his 'Aitareyalochanam.' The Adhikarana starts thus 'सहस्र संवत्सरं तद् आयुषां असम्भवात् मनुष्येषु।'.....Purva Mimansa, 6-7-12. The discussion was like this. The question of thousand solar or lunar years was excluded—the impossibility of a man living up to that age. With the help of special medicines (Rasayana) man may live up to that age—but that is also excluded for want of positive proof—moreover the Sruti says 'शतायुर्वै पुरुषः।'—'men live up to hundred years.' Then it was assumed that thousand years means thousand months. This was also excluded for a man has then to start a 'yajna' from his boyhood. The last conclusion arrived at was that thousand years (सहस्र संवत्सर) means 'thousand days'!

The fact is later Mimansakas, lexicographers, physicians, etc., not possessing a clear knowledge of the astronomical truths began to discuss and sometimes to change the original texts, thinking thereby they were correcting the same. "The only other authority I can find for supposing that Phalguna was a Vasanta month is the statement of Susruta's medical work, that Phalguna and Chaitra make Vasanta. But on a closer examination of the passage wherein this sentence occurs, it will be found to bear on its face the marks of later insertion. There are two consecutive paragraphs in Susruta, each enumerating and describing the seasons of the year. The first states that

'There the twelve months, beginning with Magha, make six seasons comprising two months each. They are Sisira (Winter), etc.....' The second paragraph then begins with the words, 'But here' (better translation would be—but at present) and continues to state 'the six seasons are Varsha (Rains), Sharad, Hemanta, Vasanta, Grishma and Pravrish' thus altogether dropping Sisira... The paragraph then proceeds to assign the months to the seasons as follows—Bhadrapada and Aswina is Varsha...and Phalgun and Chaitra is Vasanta.....' तत्र माघादयो द्वादशमासा हिमासिकम् ऋतुं कृत्वा षड् ऋतवो भवन्ति । ते शिशिरवसन्तग्रीष्मवर्षाशरदुद्हेमन्ताः ।इदं ऋतुं वर्षाशरदुद्हेमन्तवसन्त-ग्रीष्मप्रातृषः षड् ऋतवो भवन्ति ।ते तु भाद्रपदाद्येन हिमासिकेन व्याख्याताः तद् यथा । भाद्रपदाश्वयुजौ वर्षा कार्तिक-मार्गशीर्षौ शरत् । फाल्गुनचैत्रौ वसन्तः ।'

The fact is, it was not 'तत्र माघादयो द्वादशमासा, etc.,' but 'मार्गादयो द्वादशमासा, etc.,' and the first season was Vasanta. In the usual order it means that Margasirsa and Kartika is Vasanta. But owing to the falling back of the seasons it was found after a long time that Magha and Phalgun indicated Sishira, so the change was calmly introduced. In the next passage some one recently perceiving that Phalgun and Chaitra indicates Vasanta introduced that smoothly, dropping altogether Sisira which should have been the months Phalgun and Chaitra. Vaisakha and Jaisthya would then indicate Vasanta months. We have in the last passage clear indications of Spring in Vaisakha and Jaisthya, later in Phalgun and Chaitra, clear effects of "Recession" of equinoxes." A similar trouble occurred in the Amarakosha. "द्वौ द्वौ माघादिमासौ स्वाद् ऋतुः तैरयनं त्रिभिः ।" The exact reading was "द्वौ द्वौ मार्गादि मासौ, etc." Raghunath Chakravarty, an ancient Bengali commentator of Amarakosha, observed this reading but he started defending the former thus : 'मार्गादि मासौ इति उच्यमाने हि हेमन्तशिशिरवसन्तौ स्त्रिभिरयनं उक्तं

स्यात् । etc.' He doubted, if Agrahayana and Pausha began the seasons then those months indicating Hemanta in his time would start an Ayana. But Uttarayana happens in Magha 'so माघादि मासौ, etc.' is correct. In another passage in the Amarakosha we have 'षड् अमो ऋतवः पुंसि मार्गादीनां युगेः क्रमात् ।' Here there being no trouble about stating the 'ayanas' the original reading was kept. In all the Vedic literature seasons begin in the order of Vasanta, Grishma, etc.¹ All these point clearly to the fact that at one time Agrahayana and Kartika indicated Vasanta, and Vasanta, Grishma and Varsa (from vernal to autumnal equinox) indicated Uttarayana—sun's passage north of the equator. From about the time of the Vedanga Jyotisha (1400 B. C.) the sun's turning north, back from the winter solstice began to be regarded as Uttarayana. Later on when half a revolution was completed (4000 B. C.), 1st of Agrahayana indicated autumnal equinox, consequently the last day of Magha indicated winter solstice.² Bhishma's departure during Uttarayana in Magha about B. C. 3100 may be verified by any one. Lots of changes were introduced at this time. This will be clear on consideration of the following Slokas in the Vedanga Jyotisha. 'घर्महृदिः अपांशः क्षपाकासः उदग्गतौ । दक्षिणेतौ विपर्यासः षण्महर्ष्यनेनतु ।' The simple meaning of which is that when the sun is to the north of the equator, perspiration increases, there is an abundance of water and the night is shorter (than twelve hours). But when Uttarayana was made to begin from the winter solstice, the word 'घर्म' was given the meaning 'day' and 'अपांशः' 'a measure of time.' The latitude of the place where the

¹ 'वसन्तोशीर्षीवर्षाः । ते देवा ऋतवः —ब्रह्मैव वसन्तः ।...'—शतपथ ब्राह्मण—२-१-३ ।

'सुखं वा एतद् ऋतूनां यद् वसन्तः ।'—Tait. Br., 1-2-6.

'मधुय माधवथ वासन्तिकौ ऋतु अग्नेः अन्तःश्रेष्ठोऽसि...'—Yajur Veda, 13-25.

² Vide table in 'Libration of Colures,, etc.,' Journal of the Department of Letters, Vol. V.

difference between the mean half-day (twelve hours) and the solstitial day or night is 6 Muhurta ($4^h 48^m$) is $53\frac{1}{2}^\circ$. Whereas if 6 Muhurta is the difference between the length of the solstitial day and night then the latitude of the place is 35° . Similarly 'प्रपद्यते श्रविष्ठादौ सूर्याचान्द्रमसौ उदक्। सार्पाधिं दक्षिणार्कसु माघश्रावणयोः सदा ॥' originally meant that the sun reaches the north (the northern limit) when he is in the beginning of Sravistha—indicating summer solstice (the rainy season, *vide* discussion on Sravistha before). But when half a revolution was completed the Sloka was given the meaning that the sun turns to the north (having reached the southern limit) when he reaches Sravistha.

There are numerous other statements scattered throughout the ancient literature which indicate that the Aryan civilization dates back at least from 16000 B.C. For all seekers after truth this will be evident without doubt.

"These points, however, are not sufficiently soaring for the Orientals. Sir William Jones says 'I engage to support an opinion which the learned and industrious M. Montucla seems to treat with extreme contempt, that the Indian divisions of the zodiac was not borrowed from the Greeks or Arabs but having been known in this country from time immemorial, and being the same in part with that used by other nations of the old Hindu race, was probably invented by the first progenitors of that race before their dispersion.' He continues, 'Now I undertake to prove that the Indian Zodiac was not borrowed mediately or directly from the Arabs or Greeks and since our solar division of it in India is the same in substance with that used in Greece, we may reasonably conclude, that both Greeks and Hindus received it from an older nation, who first gave names to landmarks of heaven and from whom both Greeks and Hindus, as their similarity

in language and religion fully evinces had a common descent.' The same subject was inquired into by the late accomplished Mr. Colebrooke who was perhaps more competent to the task than any of his contemporaries. 'I apprehend,' says this gentleman, 'it must have been the Arabs who adopted, with slight variation, a division of the Zodiac familiar to the Hindus.....' But through all the enquiries, the Hindu Zodiac claims the highest antiquity, assuming that of Egypt of which we shall presently speak to have been borrowed therefrom....."—Smyth's Cycle, Vol. I, pp. 444-445. "The Hindu claims to antiquity stands on a higher ground, as a curious but involved historical problem; and without joining the partizanship of Bailly or Delambre, Playfair or Leslie, I cannot but be somewhat influenced by the learning and sagacity of Sir W. Jones; the science and judgment of! C. Colebrooke.....; the persevering spirit of inquiry of Mr. Davis; and the talents of Schlegel. The epoch claimed for the conjunction of the planets, as the opening of their Tables of Trivalore, in the beginning of the Kali Yuga, the present Iron or corrupt age (17th Feb. B. C. 3102) may be fanciful enough; On the whole we must allow that the early Hindus applied intensely to pure sciences, since they could compute the mean motions and true places of the planets, and calculate lunar and solar eclipses; they understood the astronomical sphere, and its circles, suspected a libration of the equinoctial points, had a glimpse of geometry, was expert in instrumental observations, and enriched science with those powerful organs—arithmetic and algebra."—*Ibid*, p. 5.

In fact the Aryans, even of the Vedic age were well acquainted with the divisions of the Zodiac and numerous other astronomical truths. 'M. Mollen has shown, that there is no reason to suppose the zodiacal divisions

were unknown to the Hindus at the probable date of the Vedas.'—Memoirs de l'Academie des Inscriptions premiere serie, Vol. 3—quoted by Wilson in his translation of the Rig Veda. 'Prof. Ludwig goes further and holds that the Rig Veda mentions the inclination of the ecliptic with the equator (I-110-2) and the axis of the earth (X-86-4).'—Tilak's Orion. In the fortieth hymn of the fifth mandala of the Rig Veda we find the description of a total solar eclipse. Therein we find Atri using a Turiya Jantra (a quadrant) in locating the sun totally eclipsed '... गूढं सूर्यं तमसा अपव्रतेन तुरीयेण ब्रह्मणा अविन्दत् अग्निः ।' (Rig Veda, V, 40-6.) The sun hidden through darkness was located by Atri with the help of the movement of (his) big quadrant.' The late Mr. Tilak rightly interpreted 'turiya' to mean 'turya' or 'turiya' jantra, of the ancient astronomers. Turiya is derived from चतुर—four; hence तुरीय—तूर्य is the fourth part of the circle—a quadrant—now-a-days we have the sextant more generally used. तुरीयेण ब्रह्मणा does not mean by the action of the turiya as suggested by the late Mr. Tilak but simply 'ब्रह्मणा (वृहता-वृहत्त्वात् ब्रह्म-वृह-मन्) by a large, तुरीयेण (तूर्यं यन्त्रेण) quadrant. व्रतेन here means by the action or by the movements (requiring diligence). The ancients used the quadrants for astronomical observations. 'Meridional observations of particular celestial bodies can only be taken at intervals averaging about a day (exactly a sidereal day for a star) and it may happen that the sky is cloudy when the body to be observed is on the meridian; or the passage of the meridian may occur in the day time, which for other bodies than the sun and the moon renders observation impossible. Hence such instruments as the astrolabe and parallactic instruments were not displaced by meridional instruments even when good clocks made these more effective than they can be without the power of measuring

time effectively. Accordingly we find Tycho Brahe, whose instrumental arrangements represent about the highest development which observational astronomy attained before the telescope, employed extra meridional instruments. Among these were quadrants which could be moved in azimuth—that is, could sweep the horizon. Of these Tycho possessed several, the largest being six feet in radius (reading to quarter minutes of arc). (Here follow a sketch and description of the same.) Fig. 13 represents Tycho's largest extra meridional quadrant. It serves to give a good idea of the way in which observations were made before the telescope was applied to astronomical work.'—Proctor, *Old and New Astronomy*, sections 84-85.

It is worth while reading the article on 'Telescope' (Enc. Brit. 10th Ed.) regarding whether the ancients really knew the telescope and its uses. Assuming the ancients had no telescope, it does not follow that astronomical observations by them were inaccurate. We have a recent example in Hevelius—the great German astronomer to the point.

'The instruments used by Hevelius represented the last attempts of pre-telescopic astronomy to secure accuracy of observation. He used the telescope to obtain views of the heavenly bodies, but, he claimed that for determining position, pin-holes were better than telescopes, and throughout his observing career, he refused the assistance which the magnifying power of the telescope would have given him.'—Proctor, *Old and New Astronomy*, section 87.

'In 1673 Hevelius published the first part of his '*Machina Caelestis*' as a specimen of the accuracy of his instruments and observations. The next year Hooke, who was ever ready to break a lance, published some observations on the '*Machina Caelestis*,' in which he treated the author with great illiberality. Such was the

interest taken in the controversy that Halley was deputed by the Royal Society to repair to Dantzic and investigate the subject of dispute. Halley arrived at the house of Hevelius on the 26th of May, 1679, and remained with him till the 17th of July following. On the evening of his arrival, the two astronomers commenced observing, Halley with the telescope, and Hevelius with plain sights (*nudis oculis*) to ascertain, from a comparison of the observations, which of the two methods gave the most correct results; and such was the dexterity which Hevelius had acquired aided by a most excellent eye-sight, that the difference of their readings seldom amounted to more than a few seconds, and in no case to so much as a minute. Halley therefore decided against our countryman.'—Smyth, Cycle, Vol. I, p. 49. We had such an example in Chandra Sekhara, the astronomer of Orissa.

From the evidences produced before, it will be perfectly clear that all doubts of placing the Aryan civilization beyond 16000 B. C. must vanish. The great French astronomer, M. Arago was convinced of the highest antiquity (13000 B. C.) of the Egyptian zodiacs. "...Such is the exposition of which there are now several editions in circulation; it seems a fac-simile of that given out by Remi Raige at the commencement of the present century, but derives its claim to this notice, from bearing the valuable stamp of M. Arago. Yet it requires a fuller explanation than is given, since the act of connecting the summer solstice with Capricorn, the winter one with Cancer, the vernal equinox with Libra, and the autumnal one with Aries, is in opposition to the generally received opinions. If the scheme is applicable to the nocturnal risings of the constellations, the date may be about B. C. 762, when the first erratic Egyptian month, Thoth, coincided with Pisces; but if the more usually received matutinal risings of the signs are to be understood,

then will the date ascend to 13000 years earlier, or half a revolution of the equinoxes and solstices. But the enormous length of such astronomical periods, while elevating the mind to the loftiest conceptions of which it is capable, is also a check upon hasty conclusions. Though no subject seems more completely beyond human powers than celestial mechanics, no other has attained a nearer approach to perfection; and the motions of the heavenly bodies have been so perseveringly scrutinised, that time has become rather a mere relation than an agent in the inquiry; a result and not a principle.'—Smyth, *Cycle*, Vol. I, p. 457. But there is no danger of hasty conclusions now. The importance of Abhijit, the southing and disappearance of Agastya (Canopus), his reappearance and the religious observances connected with it, the history of lots of other nakshatras clearly evince the extreme antiquity of the civilization of the Aryans before their dispersal and recent conversion into Christianity, Mahometanism, etc. Let us all ponder over the numerous evidences left by our great ancestors scientifically and then we shall not hesitate a moment in declaring that ours is a civilization as ancient as any human record may show.

In conclusion, I express my heartfelt gratitude to the Hon'ble Sir Asutosh Mookerjee, Kt., C.S.I., for his deep sympathies and kind permission to publish this paper in the *Journal of the Department of Science, Calcutta University*.

ADDENDA

1. In support of the derivation of the word 'नक्षत्र' from नक्ष and त्रै (p. 2) cf—'...नक्षत्राणाम् अहम् शशी' (Gceta, Ch. X. 21) 'I am the moon among the nakshatras,' i.e., the moon is the chief nakshatra.

2. (Pp. 32-34) The statement of the Saptarshis being in Magha will be satisfactorily understood when we remember that 'सप्तर्षयः' is a synonym of 'रश्मिः,' 'the rays of the sun'—*vide* Yaska's Nighantu. At the time of Yudhisthira the rays 'सप्तर्षयः' of the sun were in Magha, i.e., the rays were maximum when the sun was in Magha. The maximum happens at the summer solstice. Hence the Saptarshis were in Magha at the time of Yudhisthira meant that the summer solstice had entered the asterism. The opposite phenomenon of the rays 'गावः' of the sun being almost powerless 'हन्यन्ते' in Magha 'मघासु' (*vide* pp. 22-23) happened half a revolution earlier.

3. In the map of the north polar region of the heavens, the small diamond-shaped figure together with the other one nicely represents a hammer-headed shark—*Zygaena Blochii*—one of the three species, one found in the Indian Ocean and the other two in the tropical and temperate seas. *Fide* Francis Day's 'Fishes of India,' Vol. II, p. 719, plates CLXXXIV—fig. 4, CLXXXVI—fig. 4, CLXXXVIII—fig. 4, also plate in Webster's International Dictionary. I am indebted for this suggestion to Prof. Ekendra Nath Ghosh, M.Sc., M.D., of the Medical College, Calcutta. If a hammer-headed shark was meant to be the *sisumara* by the ancients then the stanzas quoted *re* the same have to be explained differently. Uttanpada will then be a star in the small lozenge-shaped figure. From Bhaskara's notes however, it clearly follows that he took the *sisumara* to be the porpoise.

4. (Pp. 40-41) M. Bailey in his 'Traite de l'Astronomie Indienne et Orientale, p. 144, calculated the amount of precession in longitude (according to modern formula) in 4,801 years from A.D. 1700 to B.C. 3102 and got $2^{\circ} 7' 8'' - 1^{\circ} 45' 22''$ or $1^{\circ} 51' 17''$. The amount $1^{\circ} 45' 22''$ or $1^{\circ} 51' 17''$ is the correction for precession for 4,801 years according to Lagrange or Laplace's formula, as we go back to antiquity from 1700 A.D. The mean annual rate of precession is then $48.9''$.

5. (P. 42) In the calculation of the date of the Mahabharata war from Abul Fazl's figures, Breunand consulted Gladwin's translation of the Ain-I-Akhari, where the dates are wrongly translated. Jarrett's translation of the same reads thus 'Towards the close of the Dwapara Yuga, 135 years before the beginning of the Kali Yuga and 4,831 years anterior to this the 10th of the Divine era, this event (Mahabharata War)....' Thus the beginning of the Kali Yuga is 3102 B.C.

6. The significance of the terms आदि, मध्य and अन्त, (e.g., कर्कटादि, मृग or मकरादि, मेषादि, मीनान्त, अश्विनादि, सर्पार्ध, i.e., middle of Aslesha, etc.) should be noted carefully. The first, middle or end portion of a *rasi* or *nakshatra* may mean any point within the first, second or last third of the same. If *adi* indicates the very first point, then a certain phenomenon is to be restricted only to a particular year. Whereas the equinox or the solstice in the *adi* or the *anta* of a *rasi* or *nakshatra* has in reality the range of about 700 or 300 years respectively in each case.

ZOOLOGY

Prepotency of Stimuli.

A Study in the Behaviours of House Flies.

By

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When a number of forces act, upon an organism, its orientation is determined by their relative intensity; *i.e.*, the action of one form of stimulus influences the action of another acting simultaneously on the animal, and the form of the response given by the animal will be modified in answer to the prepotent stimulus. It is evident that of the two stimuli one suppressing the other must be stronger. But this does not clear up the problem. What is the criterion of intensity in such a case? It has been suggested that the reaction of an organism in cases of multiple stimuli is determined by the degree of their vital importance. We read, "Wherever we find that one class of stimuli regularly yields to another if the two act together, it is safe to assume that the prepotent stimulus is more important to the organism's welfare than the vanquished one."¹ This statement, however, cannot be accepted as a universal statement. There are cases where it has been found that the food-taking instinct is overcome by the photic. In order to determine the relation between the food substance and stimuli of certain other class, I arranged the following experiments.

¹ The Animal Mind by Washburn, p. 288.

I. OBSERVATIONS OF RANDOM CASES.

I. FOOD *versus* TRANSPARENT SOLIDS.II. FOOD *versus* DARKNESS.

(i) *Food versus Transparent Solids.*—The flies were first offered various food substances, such as, honey, sugar, meat-juice, etc. They were perfectly at rest enjoying their food. Then they were approached with a transparent glass beaker from above. Attempts were made to shut them up in the beaker; and in majority of cases the approach of the beaker had no influence whatsoever upon the pre-existing stimuli. That is, their food-taking behaviour was in no way altered and they allowed themselves to be confined with all easiness, as is evident from the readings below. It should be noted here that in order to avoid any shadow of the glass which would present altogether different problems, that is, the skioptic problems, all the observations were made in the shade.

+ = Confinement.

0 = Flight.

1st set of 100 readings.

1	+	11	+	21	+	31	+	41	0	51	+	61	+	71	+	81	+	91	+
2	+	12	+	22	+	32	+	42	+	52	+	62	+	72	+	82	+	92	+
3	+	13	0	23	+	33	+	43	+	53	+	63	+	73	+	83	+	93	+
4	+	14	0	24	+	34	+	44	+	54	+	64	0	74	+	84	+	94	+
5	+	15	0	25	+	35	+	45	+	55	+	65	+	75	+	85	0	95	+
6	+	16	+	26	+	36	+	46	+	56	+	66	+	76	+	86	+	96	+
7	0	17	+	27	+	37	+	47	+	57	0	67	+	77	+	87	0	97	+
8	+	18	+	28	+	38	0	48	+	58	+	68	+	78	+	88	0	98	+
9	+	19	+	29	+	39	+	49	+	59	+	69	+	79	+	89	+	99	0
10	+	20	+	30	+	40	+	50	+	60	0	70	+	80	+	90	+	100	+

Here 87 of the 100 individuals did not respond to the stimulus of the transparent solid, *i.e.*, the latter had no modifying effect upon the former.

2nd set of 100 readings.

1	+	11	+	21	+	31	0	41	+	51	+	61	+	71	+	81	+	91	0
2	+	12	+	22	+	32	+	42	+	52	+	62	0	72	+	82	+	92	0
3	+	13	+	23	+	33	0	43	+	53	+	63	+	73	+	83	+	93	+
4	+	14	+	24	+	34	+	44	+	54	+	64	+	74	+	84	+	94	+
5	+	15	+	25	+	35	+	45	+	55	+	65	+	75	+	85	0	95	+
6	+	16	+	26	+	36	+	46	+	56	+	66	0	76	+	86	+	96	+
7	+	17	+	27	+	37	+	47	+	57	+	67	+	77	+	87	+	97	+
8	+	18	+	28	+	38	0	48	+	58	+	68	0	78	+	88	+	98	+
9	+	19	+	29	+	39	+	49	+	59	+	69	+	79	+	89	+	99	+
10	+	20	+	30	0	40	+	50	+	60	+	70	0	80	+	90	0	100	+

In this series 88% allowed themselves to be confined, *i.e.*, did not respond at the approach of the glass.

3rd set of 100 readings.

1	+	11	+	21	+	31	0	41	+	51	+	61	+	71	+	81	+	91	+
2	0	12	+	22	+	32	0	42	+	52	+	62	+	72	+	82	+	92	+
3	+	13	0	23	+	33	0	43	+	53	+	63	+	73	+	83	+	93	+
4	+	14	+	24	+	34	+	44	+	54	+	64	+	74	0	84	+	94	+
5	0	15	+	25	+	35	+	45	+	55	+	65	+	75	+	85	+	95	+
6	0	16	+	26	0	36	+	46	+	56	+	66	+	76	+	86	+	96	+
7	+	17	+	27	+	37	+	47	+	57	+	67	+	77	+	87	+	97	+
8	+	18	+	28	+	38	+	48	+	58	+	68	+	78	+	88	+	98	+
9	0	19	+	29	+	39	+	49	+	59	+	69	+	79	+	89	+	99	+
10	+	20	+	30	+	40	+	50	+	60	0	70	+	80	+	90	+	100	+

Here the food stimulus in 89 cases out of 100 did not yield to the solid.

4th set of 100 readings.

1	+	11	+	21	+	31	+	41	+	51	+	61	+	71	+	81	0	91	+
2	+	12	+	22	+	32	+	42	0	52	+	62	+	72	+	82	0	92	+
3	+	13	+	23	+	33	+	43	0	53	+	63	0	73	+	83	+	3	+
4	0	14	+	24	+	34	+	44	+	54	+	64	0	74	0	84	+	94	+
5	+	15	0	25	0	35	+	45	+	55	+	65	+	75	+	85	+	95	+
6	+	16	0	26	0	36	+	46	0	56	+	66	0	76	+	86	+	96	+
7	+	17	+	27	+	37	+	47	+	57	+	67	+	77	+	87	+	97	+
8	+	18	0	28	0	38	+	48	+	58	+	68	+	78	+	88	+	98	+
9	+	19	+	29	+	39	+	49	+	59	+	69	+	79	+	89	+	99	+
10	+	20	0	30	0	40	+	50	+	60	+	70	+	80	+	90	+	100	0

No effect of solid in 81% of cases.

5th set of 100 readings.

1	+	11	+	21	+	31	+	41	+	51	+	61	+	71	+	81	+	91	+
2	0	12	+	22	+	32	0	42	+	52	+	62	+	72	+	82	+	92	
3	+	13	+	23	+	33	+	43	+	53	+	63	+	73	+	83	+	93	+
4	0	14	+	24	+	34	0	44	+	54	+	64	0	74	+	84	+	94	+
5	0	15	+	25	+	35	+	45	+	55	+	65	+	75	+	85	+	95	+
6	+	16	+	26	+	36	+	46	+	56	+	66	+	76	+	86	+	96	+
7	0	17	+	27	0	37	0	47	+	57	+	67	+	77	+	87	+	97	+
8	0	18	+	28	+	38	+	48	+	58	+	68	+	78	+	88	+	98	+
9	+	19	0	29	0	39	+	49	+	59	+	69	+	79	+	89	+	99	+
10	+	20	+	30	+	40	+	50	+	60	+	70	+	80	+	90	+	100	+

87% did not yield.

It is interesting to note that the range of variation is from 81 to 89 and except in the 4th set the variation is very slight, it being by only one.

(ii) *Food versus Darkness*.—The second series of observations were made in order to determine the effect of darkness on the food-taking behaviour of the house fly. Here attempts were made to cover the animals with a metal bowl. Consequently, when it was lowered down around an individual already at rest and partaking its food, the effect was quite darkness. It may be questioned, whether the reactions to such a stimulus was one due to the opaque solid or the darkness produced by it. From other observations it has been ascertained that the approach of an opaque, solid mass without producing darkness does not modify the food-taking instinct of flies. Therefore, the behaviours as represented in the following tables may safely be considered as reactions to darkness which overcomes the food-taking reaction.

1st set of 100 readings.

1	0	11	0	21	0	31	0	41	0	51	0	61	0	71	0	81	+	91	0
2	0	12	0	22	0	32	0	42	0	52	0	62	0	72	0	82	0	92	0
3	0	13	0	23	0	33	+	43	0	53	0	63	0	73	0	83	0	93	0
4	0	14	0	24	0	34	0	44	+	54	0	64	0	74	0	84	0	94	0
5	0	15	+	25	0	35	0	45	0	55	0	65	0	75	0	85	0	95	0
6	0	16	0	26	0	36	0	46	0	56	0	66	0	76	0	86	0	96	+
7	+	17	+	27	0	37	0	47	0	57	0	67	0	77	0	87	+	97	0
8	0	18	0	28	+	38	0	48	0	58	0	68	0	78	0	88	0	98	+
9	0	19	+	29	0	39	0	49	0	59	0	69	0	79	0	89	0	99	0
10	0	20	0	30	0	40	+	50	+	60	0	70	0	80	0	90	0	100	0

87% responded to the approach of the bowl by flight, *i.e.*, in these cases of flight darkness prevailed upon the stimulation of food.

2nd set of 100 readings.

1	0	11	0	21	0	31	0	41	0	51	0	61	0	71	0	81	0	91	+
2	+	12	+	22	0	32	0	42	0	52	0	62	0	72	0	82	0	92	0
3	+	13	0	23	0	33	0	43	0	53	+	63	0	73	0	83	9	93	0
4	0	14	0	24	0	34	0	44	0	54	0	64	0	74	0	84	0	94	0
5	0	15	0	25	0	35	+	45	0	55	0	65	0	75	0	85	+	95	0
6	0	16	0	26	0	36	0	46	0	56	0	66	0	76	0	86	0	96	0
7	0	17	0	27	0	37	0	47	0	57	+	67	0	77	0	87	0	97	0
8	0	18	0	28	0	38	0	48	0	58	+	68	0	78	0	88	0	98	0
9	+	19	+	29	0	39	0	49	0	59	0	69	0	79	0	89	0	99	+
10	+	20	0	30	0	40	0	50	0	60	+	70	0	80	0	90	0	100	0

Prevalence of darkness upon food in 86 % of cases.

3rd set of 100 readings.

1	+	11	0	21	0	31	0	41	0	51	0	61	0	71	0	81	0	91	+
2	0	12	0	22	0	32	0	42	0	52	0	62	0	72	0	82	+	92	+
3	0	13	0	23	0	33	0	43	0	53	0	63	0	73	+	83	0	93	0
4	0	14	0	24	0	34	0	44	0	54	+	64	+	74	0	84	0	94	0
5	0	15	0	25	0	35	0	45	0	55	0	65	0	75	0	85	0	95	0
6	0	16	0	26	0	36	+	46	0	56	0	66	0	76	0	86	0	96	0
7	0	17	0	27	0	37	0	47	0	57	0	67	0	77	+	87	0	97	0
8	0	18	0	28	0	38	0	48	0	58	0	68	0	78	0	88	0	98	0
9	0	19	0	29	+	39	0	49	0	59	0	69	0	79	0	89	0	99	0
10	0	20	0	30	0	40	0	50	+	60	+	70	0	80	0	90	0	100	0

In 88% of cases food stimulation yielded to darkness.

4th set of 100 readings.

1	0	11	0	21	+	31	0	41	0	51	0	61	0	71	0	81	0	91	0
2	0	12	0	22	0	32	0	42	0	52	0	62	0	72	+	82	0	92	0
3	+	13	0	23	0	33	+	43	0	53	0	63	0	73	0	83	0	93	0
4	0	14	0	24	0	34	0	44	0	54	0	64	0	74	0	84	0	94	6
5	0	15	0	25	0	35	0	45	0	55	0	65	0	75	+	85	0	95	0
6	0	16	0	26	0	36	0	46	0	56	0	66	0	76	0	86	0	96	+
7	0	17	0	27	0	37	0	47	0	57	0	67	0	77	0	87	0	97	6
8	+	18	0	28	0	38	+	48	0	58	0	68	0	78	+	88	0	98	0
9	0	19	0	29	0	39	0	49	0	59	0	69	0	79	0	89	0	99	0
10	0	20	0	30	0	40	0	50	0	60	+	70	0	80	+	90	0	100	0

Yielding of food stimulus to darkness in 89% of cases.

5th set of 100 readings.

1	0	11	0	21	+	31	0	41	0	51	0	61	0	71	0	81	0	91	0
2	0	12	+	22	0	32	0	42	+	52	0	62	0	72	0	82	0	92	0
3	0	13	0	23	0	33	0	43	0	53	+	63	0	73	0	83	0	93	0
4	0	14	0	24	0	34	+	44	0	54	0	64	0	74	0	84	0	94	0
5	0	15	0		0	35	+	45	0		0	65	+		0	85	0	95	0
6	+	16	0	26	0	36	0	46		56	0	66	0	76	+	86	0	96	0
7	+	17	+	27	0	37	0	47		57	0	67	0	77	0	87	0	97	+
8	0	18	0	28	0	38	0	48		58	0	68	0	78	0	88	0	98	0
9	+	19	0	29	0	39	0	49		59	0	69	0	79	+	89	0	99	0
0	0	20	0	30	0	40	0	50	0	60	+	70	0	80	0	90		100	0

In 80% of cases darkness prevailed upon the stimulus of food.

The range of prepotency of darkness varies from 80 to 89. Here too, excepting the last table, the degree of variation is slight.

II. THE STUDY OF INDIVIDUAL FLIES.

The following table shows the modification of food-taking behaviour of one and the same individual fly by a transparent solid as well as darkness.

	Glass.	Metal.		Glass.	Metal.		Glass.	Metal.		Glass.	Metal.		Glass.	Metal.		Glass.	Metal.
1	+	0	14	+	0	27	+	0	40	+	0	53	+	0	66	+	0
2	+	0	15	+	+	28	+	+	41	+	0	54	+	0	67	+	0
3	+	0	16	+	0	29	+	0	42	+	0	55	+	0	68	+	0
4	+	0	17	0	0	30	+	0	43	+	0	56	+	0	69	+	0
5	+	0	18	+	+	31	+	0	44	0	0	57	+	0	70	+	0
6	+	0	19	0	0	32	+	+	45	+	0	58	0	0	71	+	0
7	+	0	20	+	0	33	+	0	46	+	0	59	+	+	72	+	+
8	+	0	21	+	0	34	+	0	47	+	0	60	0	0	73	+	0
9	+	+	22	+	0	35	+	0	48	+	0	61	+	0	74	+	0
10	+	+	23	+	+	36	+	0	49	+	0	62	+	0	75	+	0
11	+	0	24	+	0	37	+	0	50	+	0	63	+	0	76	0	0
12	+	0	25	+	0	38	+	0	51	+	0	64	+	0	77	+	0
13	0	0	26	0	0	39	+	0	52	+	0	65	+	0	78	+	0

By an analysis of this table, it is noticed that the percentage of the individuals that did not alter their food reaction at the approach of the glass beaker but reversed it when darkened is 81. If we take the percentage of those cases alone where darkness decidedly prevails upon food stimulus irrespective of any influence of the transparent solid upon it (*i.e.*, food), we find that the number of flies reversing their food-taking instinct in response to darkness, go still higher than in the preceding

tables, the proportion being 91 to 86. The latter number is the mean percentage of the readings concerning Food *versus* Darkness.

Are we safe in our assertion that this yielding of food stimulation to darkness is of more vital significance than the food itself? All the behaviouristic peculiarities of animate forms do not always bear upon their organic welfare. It will be too radical a remark to say, for instance, that certain flies are directed towards the source of dazzling light in preference to drops of honey or some other substance which they have been found enjoying under other circumstances, because they are more benefited by jumping in the fire than by sucking honey. For the solution of the perplexing problems of behaviour we must wait till we gain an insight into the physiology of irritability.

Reversal of Thigmotropism in Hydra

By

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I

INTRODUCTION.

It is found, in course of our attempts to discover the causal mechanism of animal activities, that there is a fundamental relationship between an organism and a certain set of stimuli. The whole organic world is attuned with the conditions of its environment, internal as well as external, and the activities of animals are shaped out by the acting forces. With the changed conditions of the surroundings the substance of the organism itself changes; and the result is that it behaves in a particular way. It has also been brought to light that different organisms follow different methods of behaviour when subjected to one and the same influencing factor. The causal mechanism hitherto unpenetrated of such relationism has been given the name "irritability" and the influences, the "stimuli." However, due to this irritability there must be some sort of impulsion in the protoplasm, *i.e.*, the organic substance, which seeks expression and does find its way out in the fulfilment of a particular act, and the animal is said to "behave."

Complex behaviours of adult animals have evolved from the simple and random discharge of their primitive impulse. Facts lead us to the conclusion that the origin of all these relational phenomena or tropism is not necessarily concerned in the organic welfare of organisms. They are not always adapted to meet the same organic end. Out of random courses of expression only those that are useful to the individual are selected by nature and fixed in the heredity of the animal; and thus they are rendered into a racial inheritance. In spite of specific irritability which alone is responsible for tropic relations, they are not fixed once for all. Organic evolution will be an unmeaning phraseology, if we fail to establish a course of evolution in the realm of animal behaviour *pari passu* with the evolution of structure; because an organism is not a mere structure of variety of tissues. There has always been along with the variation of tissue, something else which has shared, perhaps, a greater bulk of responsibility to bring the animal kingdom where it is in the present moment. This is variability of tropism leading to diverse modes of behaviour. Thus the range of adaptation is widened and the chances for natural selection to come across a greater amount of material to work with are enhanced.

The more we acquaint ourselves with the facts of animal behaviour the more it becomes clear to us that there are different degrees of plasticity of instinct varying from zero to one of high complexity. It will be manifested to us that various life processes, *viz.*, the phenomena of reversal of instinct, conation, experience and practical judgment and so forth are evolved in different groups of animals in demand of the situation created by their environment. All these processes are inter-related and contributive towards mental complexity.

It seems to us that plasticity of instinct begins with its reversal, that is, the animal's capacity to change the mode of its reaction to a particular stimulus under altered circumstances. For illustration it may be cited here the phototropic reactions of *Polygordius* and certain Copepods. These creatures are according to Loeb negatively photo-tactic under normal conditions; but an increase of the salt content of the sea-water made them positive. (*Vide* The Evolution of Animal Intelligence by Holmes, p. 49.) A good collection of cases of reversal of tropisms has been embodied in Holmes' "Studies in Animal Behaviour." Reversal of phototropism has been more satisfactorily and extensively studied than other tropisms. Very little has been achieved in the field of thigmotropism or reaction to contact. Moreover, what has been done up till now in connection with the reversal phenomenon has been approached with a view to generalisation. But by taking this method of procedure one "Soon finds himself hampered by the paucity of knowledge in those fields to which he would naturally turn for helpful suggestion." In order to trace the course of behaviouristic evolution we must accumulate more facts.

II

REVERSAL OF THIGMOTROPISM CAUSED BY HUNGER.

The food-taking reaction of *Hydra* is distinct from the contact reaction. In *Hydra* the food reaction is the only form of positive thigmotropism. It takes in a solid edible object, and to all other mechanical stimulation it reacts negatively. But *Hydra* is equipped with the power of change from negative to positive thigmotropism at the time of extreme hunger; and such a reversal is evidently a serviceable endowment with the animal.

First, we shall describe the behaviours of a number of individuals of the genus taken fresh from its habitat.

In following the behaviours of *Hydra* when it is approached by a mechanical stimulus, say, a tiny portion of a plant leaf which we used throughout the whole course of our experimentation, it is noticed that a set of movements, sometimes quite complicated, of the animal body is manifested. In spite of our attempt at a chemical interpretation of animal behaviours it cannot be readily ascertained whether the reaction is of a positive or negative nature in cases where varied movements are evolved before the organism is finally oriented in relation to the stimulus applied, and, therefore, when we do not get any direct response on the part of the organism analogous to the phenomena of chemical attraction and repulsion. Only close observation of these movements and their duration will furnish us with a clue to the nature of relation established between the stimulus and its subject. Hence, we give here a detailed description of the behaviours of the *Hydra* studied in this connection so that the co-workers in the field may draw their own conclusion.

REACTION OF FRESH INDIVIDUALS TO A MECHANICAL STIMULUS.

Specimen No. I.

1. As stated above, a small portion of a plant leaf was attached to the hypostome of the animal and it behaved in this manner :—

(1) The *Hydra* became very restless with constant backward and forward movements.

(2) The body of the animal became so elongated as to render any differentiation between the tentacles and elongated body almost difficult.

(3) After a strenuous struggle for about five minutes, the animal succeeded in getting rid of the foreign material and became quiet, although it kept its body in the same elongated condition. Occasionally the animal contracted its body very slightly and bent it just above the foot.

2. The same stimulus was applied again. It was held in contact with the tentacles. The animal at once receded.

3. The Hydra was stimulated as before. At this time too the animal behaved by turning its body sideways, leaving the object of stimulus near the distal end. The stalk remained fixed on the substratum.

4. In the fourth trial the animal responded to the stimulus applied as in the preceding cases by contracting its body to an oval shape.

5. This is a case where the Hydra responded to the usual mechanical stimulus by a bend of its body along the vertical line, *i.e.*, it turned by a vault and was finally oriented as in 3.

6. Same as 3.

7. In this case too the portion of the plant leaf was placed in contact with the tentacles. The object was held for 20 seconds in the bend of one of the tentacles and then the Hydra suddenly contracted all of them finally taking the position as in 3.

8. Same as 3.

9. Same as 3.

10. The Hydra reacted to the stimulus brought in contact with a tentacle by contraction of the latter. The animal was oriented as in 3.

11. Same as 4.

12. Same as 3.

13. Same as 4.

14. At the time of applying the stimulus the animal was in contracted condition—almost ball-shaped. The

stimulus was placed in contact with the animal body at the region of the hypostome. The animal extended, all the tentacles enlongated and the anterior end of the Hydra went far beyond the source of stimulus.

15. A few of the tentacles were touched with the foreign object and the result was that immediately all the tentacles contracted.

16. The stimulus was applied as in the foregoing trial. The animal body at once contracted.

17. Here also the animal was similarly treated. It contracted to a ball and began to float with the tentacles downward.

18. The stimulus was given as before. The animal at once stood erect. The stalk was fixed to the substratum with the tentacles fully spread.

19. The stimulus was applied to a number of tentacles. Here a series of behaviours evolved.

(1) The tentacles in touch with the foreign substance first contracted and then were thrown on it. This was repeated several times. This to and fro movement of the tentacles covered a period of 2 minutes.

(2) The particle was caught in the bends of 3 tentacles and was brought upon the hypostome. In this condition, *i.e.*, with the object on the hypostome the animal remained for about a minute.

(3) At length the remaining 3 tentacles still free were engaged in gripping the foreign body. Finally the animal contracted and its position became reversed with its hypostome on the particle. This struggle continued till all of a sudden the Hydra left the object of struggle. The whole contest lasted for full 6 minutes.

It will not be out of place, perhaps, to note that this last-mentioned behaviour gives sufficient indication of the development of the process of trial and error in the Hydra.

Specimen No. VII.

1. A portion of a plant leaf was brought in contact with a tentacle of the Hydra. The tentacle was at once drawn aside.

2. The stimulus was similarly applied. Reaction was given by the animal by its immediate turning back.

3. On application of the stimulus to a tentacle, the following activities on the part of the Hydra were observed :—

(1) This tentacle, *i.e.*, the one in contact with the stimulus, with another immediately contracted, the remaining four being in an expanded condition.

(2) The anterior portion of the animal body bent away from the object.

4. Here, in the fourth trial the stimulus was applied to a number of tentacles. The Hydra reacted by a sudden vault. The body and the tentacles contracted almost to a globule.

5. The stimulus was placed in contact with one of the tentacles which at once contracted. The remaining ones contracted after a short period and the body became globular.

6. This time the stimulus was placed in the whorl of 3 tentacles. The object was rejected by a forward push with the middle tentacle.

7. The stimulus being attached to a tentacle the Hydra reacted by contracting this tentacle and another by its side. The body remained stationary.

8. Application of the stimulus was as in the preceding case. Three tentacles contracted at once while the other three remained extended. The anterior end of the body was raised and drawn aside.

9. The stimulus was applied as in No. 8. The tentacle curled away from the object.

10. Same as 9.

11. In this trial the stimulus was applied to two tentacles which at once contracted. The body also contracted.

12. Here too the stimulus was brought in contact with two tentacles which at once contracted. The anterior end of the body was raised and drawn aside.

13. The stimulus was brought in contact with one tentacle which contracted and bent away from the object. The body too contracted.

14. The stimulus was applied in the similar manner. The Hydra reacted by a sudden contraction of the tentacle touched by the object, others remaining expanded.

15. The stimulus was placed in contact with two tentacles which curled round the object and held it in the bend. Next, the object was brought in touch with the hypostome. The object was held in this condition for two or three seconds only. Then the animal receded leaving the object.

16. Now the stimulus was applied to one tentacle only. The body contracted with all the tentacles.

17. Application of the stimulus was made in the foregoing manner. The animal reacted by a vault.

18. The stimulus was applied again to one tentacle only which remained in contact with the object for half a minute and then contracted.

19. The stimulus was again applied as before. The tentacle curled round the object which was drawn near the hypostome (though not in contact with it). The animal remained in this condition for a few seconds and receded.

20. The stimulus was again applied in the similar manner. The tentacle in contact with the object contracted after half a minute. This was followed by contraction of the whole animal to a globule.

Specimen No. III.

1. The stimulus being applied to one of the tentacles the Hydra at once fled away.

2. The stimulus was applied as before. The tentacle which was in contact with the object was immediately withdrawn.

3. The stimulus was placed in contact with one of the tentacles. This tentacle was drawn towards the hypostome with another. The other tentacles also contracted after a short time.

4. The stimulus was applied again in the previous manner. The Hydra at once contracted itself and ultimately became globular.

5. The Hydra was stimulated in the same manner. This tentacle, *i.e.*, the one in contact with the object together with another suddenly contracted; but the body of the Hydra remained stationary.

6. The stimulus was applied as before. The tentacles contracted to a slight extent, then expanded and touched the object again. This was followed by immediate contraction of the whole animal which became globular.

7. The stimulus was applied in the similar process. The Hydra slowly moved away from the object.

8. The stimulus was applied as before. The Hydra at once contracted with its three tentacles which again expanded and touched the object and then contracted. This sort of operation went on for 3 times; then the whole animal became globular.

9. The stimulus was applied to one of the tentacles. The Hydra bent itself away from the object.

10. The Hydra was stimulated as before. The tentacle in contact with the object of stimulus bent away from it. The body contracted to a slight extent.

11. Same as 10.

12. This time too the stimulus was applied to one of the tentacles. This tentacle expanded and gave the object a sudden push by which the object floated away from the Hydra.

13. The stimulus was again applied in the similar manner. The tentacle which was in contact with the object remained stationary for 2 or 3 seconds and suddenly the animal contracted its tentacles with the body and became globular.

14. In this trial also the stimulus was placed in contact with a tentacle. The animal tried to bend this tentacle round the object but the object floated away from the animal.

15. The stimulus was placed near three tentacles. The animal touched the object with these three tentacles, remained as such for $\frac{1}{4}$ minute and then slowly moved away from the object.

16. The stimulus was applied to a tentacle. The animal caught the object with all the tentacles; which was drawn towards the hypostome. The animal thus remained for $\frac{1}{2}$ minute and then contracted to a globule.

17. The stimulus was applied to a tentacle. The animal remained as such for $\frac{1}{2}$ minute and slowly moved away from the object.

18. The stimulus was applied to two tentacles. The Hydra remained stationary for 1 minute and slowly contracted to a globule.

19. The stimulus was applied to three tentacles. The Hydra drew the object and placed it in contact with the mouth and remained stationary for $1\frac{1}{4}$ minutes and slowly moved away from the object.

20. The stimulus was placed in the whorl of the tentacles. At once the tentacles expanded and touched the object which was drawn to the mouth. The animal remained in this condition for $2\frac{1}{4}$ minutes and then slowly

moved the anterior portion of its body away from the object.

REACTION OF THE INDIVIDUALS TO A MECHANICAL
STIMULUS AFTER THEY HAVE BEEN SUBJECTED
TO STARVATION.

All the specimens studied in this connection were kept in distilled water without food, and their thigmotropic reaction was examined at different periods of starvation.

Specimen No. I.—Period of Starvation—3 days.

1. One of the tentacles was stimulated by the same object used in the foregoing cases.

(1) The animal remained in association with the object for two minutes involving the following activities :

(i) The tentacle expanded on the object.

(ii) The tentacle waved on the object from its one end to the other.

(iii) The tentacle was taken to a side of the object.

(iv) The tentacle touched the object several times.

(v) At all these different positions of the tentacle very vigorous attempts were noticed on the part of the animal to pull the object nearer to the hypostome.

(2) The animal slowly moved away.

2. The stimulus was placed in contact with two tentacles.

(1) One of the tentacles fully looped the object whereas the tip of the other tentacle simply touched it at a side only.

(2) The object was pulled to a certain distance towards the hypostome.

(3) At this moment the curved tips of the remaining tentacles too captured the object and dragged it around in the water.

(4) The animal then moved away leaving the object. The process covered a period of $2\frac{1}{2}$ minutes.

3. The stimulus was applied to the tips of three tentacles.

(1) The animal remained in contact with the object for two minutes involving the following set of behaviours:

(i) The remaining three tentacles also expanded and touched the object by their tips.

(ii) All the tentacles except one were after a few seconds drawn aside.

(iii) The tentacles still in contact with the object extended further and completely looped the object.

(2) The object was thus held for the rest of the period and the animal body with the tentacles contracted.

Specimen No. II.—Period of Starvation—3 days.

1. The object of stimulus was placed in contact with a number of tentacles.

(1) One of the tentacles stimulated expanded and bent towards the object. The tentacle then touched this object by its tip slightly curved. The animal remained in this posture for a few seconds.

(2) The Hydra receded with the tentacles and body extended.

2. The stimulus was placed as in 1.

(1) All the tentacles expanded and only one spread over the object which was thus held for about a minute. The tentacles moved the object to and fro from time to time.

(2) The animal receded.

3. The stimulus was placed almost near the tentacles.

(1) Two of the tentacles extended along the side of the object.

(2) The object was brought in contact with the hypostome. The Hydra kept in this posture for $\frac{1}{2}$ minute.

(3) The tentacles were bent sidewise. The hypostome also was taken aside to a certain extent but the body of the animal was kept intact.

4. The object of stimulus was placed in touch with all the tentacles.

(1) All the tentacles expanded and only one of them touched the object. After a slight touch this tentacle contracted and again expanded up to the foreign particle, *i.e.*, the object of stimulus.

(2) This to and fro movement of the tentacle was repeated for a number of times within the period of about 2 minutes.

(3) The animal suddenly contracted.

5. The stimulus was brought in contact with a tentacle while the whole body of the animal was in a contracted condition.

(1) The tentacle touched by the object expanded over it.

(2) The animal body gradually dilated.

(3) A sudden contraction of the body and the tentacles took place.

6. The stimulus was brought in contact with the body of the animal when it was still in a contracted condition.

(1) One tentacle expanded over the foreign body.

(2) The animal gradually dilated.

(3) The Hydra suddenly contracted as in the preceding case.

7. The animal body was touched by the foreign substance and the result was that it at once moved away from it.

8. The stimulus was brought in contact with a tentacle in an expanded condition.

(1) The animal with such an expanded tentacle remained quite stationary for $\frac{1}{2}$ minute.

(2) It contracted the tentacle with the object still in touch. The result was that the particle was drawn towards the hypostome. The object was thus held for $\frac{1}{2}$ minute.

(3) A sudden contraction of the body and the tentacles took place and the object was left at a distance.

Specimen No. III.—Period of Starvation—3 days.

1. A tiny portion of a plant foliage was brought in contact with a tentacle of the specimen.

(1) The tentacle at once contracted while the other tentacles remained in an expanded condition.

(2) The contracted tentacle again expanded and touched the object. A sudden contraction of the two adjoining tentacles followed and the anterior portion of the body moved away from the foreign object.

2. The object was again applied to a tentacle.

(1) It remained stationary for a few seconds.

(2) Gradually the tentacles were drawn towards the hypostome.

(3) Suddenly the Hydra moved away from the object.

3. The foreign object was brought in touch with one tentacle only. The Hydra suddenly contracted this tentacle.

4. The Hydra was stimulated as in the preceding case. The animal remained as such for a few seconds and then slowly contracted all the tentacles. Finally it shortened the body too.

5. The specimen was similarly treated with the stimulus.

(1) It remained stationary for a short while. Then all the tentacles expanded around the object which was

held in their whorl. The animal drew the object to a certain distance towards the hypostome.

(2) The Hydra remained as such for $2\frac{1}{2}$ minutes and slowly moved away.

• *Specimen No. IV.—Period of Starvation—3 days.*

The same stimulus, *i.e.*, a portion of the plant leaf was applied to two or three tentacles and the following reactions were recorded.

- (i) All the tentacles expanded and the particle remained in their whorl for 3 minutes.
- (ii) Two of the tentacles extended further on the surface of the particle and remained thus for about $\frac{1}{2}$ minute.
- (iii) The animal receded a little with the tentacles extended as before.
- (iv) The animal at once proceeded and touched the foreign body with two or three tentacles and contracted them. This process, *i.e.*, the bringing of the tentacles in contact with the substance followed by their immediate contraction, continued for 2 minutes.
- (v) The animal suddenly contracted assuming an oval shape.

Specimen No. V.—Period of Starvation—4 days.

1. The same stimulating substance having been brought in contact with the tentacles gave rise to the following reactions in the specimen :—

- (1) Two tentacles were fully elongated immediately, one above the foreign body and the other beneath it.
- (2) These tentacles held the object as if to press it hard. It was thus maintained for about a minute.
- (3) The animal left the object and slowly moved aside with the tentacles as well as the body extended.

2. Only one tentacle was stimulated.

(1) The tentacle at once extended to its full length over the particle and remained in this position for $1\frac{1}{2}$ minutes. A slight contraction of the tentacle followed and its tip was brought almost near an end of the particle.

(2) At this stage another tentacle was fully spread over the object which was then drawn nearer to the mouth.

(3) The animal moved away with the tentacles and the body uncontracted.

3. The animal came near the object of stimulus of its own accord, spread two of its tentacles as in case 2 and at once pulled the particle so hard that it directly came in contact with the hypostome.

There was no contraction.

4. The stimulus was again applied to a tentacle.

(1) The object was caught in the loops of a number of tentacles and the Hydra ran away with it to a certain distance.

(2) All the tentacles were freed except one which gradually extended over the particle. It seemed that the particle adhered to the surface of the tentacle by some adhesive secretion. With a side-way swing of the tentacle the article of prey was carried to and away from the body of the Hydra. This continued for more than a minute.

(3) The body of the animal contracted almost to an oval shape, whereas the tentacles drew themselves a half of their full length when in an extended condition.

(4) The animal left the article and slowly receded not very far from it.

5. The Hydra turned towards the object of its own accord with all the tentacles fully stretched.

(1) One of the tentacles gradually came in contact with the object. A considerable portion of the tentacular

surface remained attached to the foreign body for 10 seconds.

(2) This tentacle was slightly moved, only its tip touching the object. From the various bends and bodily rigour on the part of the Hydra it appeared that the animal made an attempt to pull the particle towards the mouth.

(3) After a short futile attempt the animal moved aside with the tentacles and the body uncontracted.

Specimen No. VI.—Period of Starvation—4 days.

1. The object of stimulus was placed in contact with the tentacles.

(1) It was further expanded beneath the object.

(2) Immediately another was sent by the side of the former. Both of the tentacles remained as such for $\frac{1}{2}$ minute.

(3) Then the second tentacle was taken out and after a few seconds the animal receded.

2. The source of stimulus was placed at a short distance.

(1) One tentacle touched the object with its bent tip which was gradually drawn back leaving the particle at a distance.

(2) Two tentacles again extended and touched the object with their tips and then slowly drew back.

(3) The same behaviour as in (2) was reiterated for 7 times in two minutes. It was twice found out that one of the tentacles touching the object at once bent towards the hypostome as if in an attempt to bring the prey in its mouth.

Explanation of this behaviour, *i.e.*, the grasping movement, is to be sought in tactile sensation. It is to be noted here that through all these processes the other three tentacles were inactive although expanded outward.

So the expansion and coming in contact with the object of the two tentacles do not seem to be an affair of chance.

Specimen No. VII.---Period of Starvation - 5 days.

1. The plant tissue was applied to the tip of a tentacle which expanded around the former and held it for about a minute in the bend. Then the Hydra moved away the tentacle.

2. The stimulus was applied as before. The reactions were as before.

3. The stimulus was applied as in the preceding cases and the following reactions were observed.

(1) The reaction was just as (3) under 2 in specimen No. VI, but the animal behaved that way only twice without any intermission. The animal drew back.

(2) After three minutes the animal proceeded of its own accord towards the object with all the tentacles extended. Two of the tentacles came in touch with the particle, went underneath it, gave it a shake and drew back.

(3) The animal also receded.

(4) Repetition of (1) after about the same interval of time; but one tentacle remained in touch with the object for $\frac{1}{2}$ minute. The animal drew away a great distance.

4. The stimulus was again brought in contact.

(1) After only 8 or 10 seconds the animal drew back.

(2) The animal immediately began to proceed (all tentacles expanded) towards the plant tissue very slowly. It took the Hydra exactly three minutes to reach the object. Only one tentacle touched it, spread along its side and remained in contact for $2\frac{1}{2}$ minutes. Then the tentacle slightly contracted and again expanded, touched

the substance and immediately contracted. Finally the animal receded.

(3) The creature proceeded again, and touched the object with all the tentacles. Only the tips came in contact. At once the animal drew back.

5. The stimulus was again applied.

The specimen at once assumed a globular shape with the tentacles contracted.

6. The stimulus was placed in contact with one tentacle.

(1) It remained stationary for one minute.

(2) All the tentacles contracted; the Hydra moved away a short distance and remained as such for $1\frac{1}{2}$ minutes.

(3) That particular tentacle expanded and touched the foreign body and remained as such for 2 minutes.

(4) All the tentacles collapsed.

Specimen No. VIII.—Period of Starvation—5 days.

1. A fine portion of a plant leaf was applied to one of the tentacles.

(1) The Hydra remained inactive for $\frac{1}{2}$ minute.

(2) Then it suddenly contracted its tentacles with the body.

(3) Then the body was extended with all the tentacles; one of them again touched the foreign particle and remained as such for a few seconds.

(4) Then the animal slowly left the foreign body.

2. The object of stimulus was again applied to one of the tentacles.

(1) The animal clutched the foreign body by means of three tentacles and drew it near the hypostome and remained thus for $2\frac{1}{4}$ minutes.

(2) The foreign body was removed away from the hypostome and again drawn near it.

(3) The above practice was repeated for 7 times.

(4) All the tentacles with the anterior portion of the body slowly moved away from the foreign body.

3. The stimulus was applied as before.

(1) Suddenly the animal contracted its tentacles.

(2) The animal retracted its tentacles in another* direction away from the foreign body.

4. The stimulus was again applied.

(1) The animal remained inactive for $1\frac{3}{4}$ minutes.

(2) The anterior portion of the animal was drawn aside slowly.

5. The foreign body was kept in touch with one of the tentacles.

(1) The foreign body was drawn towards the hypostome by two tentacles.

(2) The foreign body was kept in touch with the hypostome for 6 minutes and 12 seconds.

(3) Slowly the tentacles contracted and the whole animal moved away from the object.

Specimen No. IX.—Period of Starvation—6 days.

1. The stimulus was placed in contact with the hypostome and a number of tentacles.

(1) After a few seconds the animal drew back a short distance, immediately proceeded to the object and brought the hypostome directly in contact with it. The object was held in the whorl of four tentacles for full 12 minutes. The tentacles gave it a push from time to time. Then the animal left the object and went a short distance away.

(2) After an intermission of two minutes the Hydra proceeded again and behaved exactly in the same manner as (1) for 3 minutes and 10 seconds after which period it went away.

2. The stimulus was attached to the tip of a tentacle. The Hydra remained inactive for a few seconds and then elongated its body so that the hypostome and the tentacles went beyond the object.

Specimen Y.—Period of Starvation—6 days.

1. The stimulus (*i.e.*, the portion of a leaf) was placed at a distance.

(1) The animal approached it slowly with all the tentacles out-spread and touched it with the bent tip of a tentacle.

(2) The Hydra slowly receded.

2. The stimulus was placed as in 1.

One tentacle extended and touched the particle and suddenly contracted together with the others. The body then contracted to an oval shape.

3. The specimen was stimulated as before.

The reaction was observed as in 2.

4. Here only one tentacle was stimulated.

(1) This very tentacle extended further touching a side of the object.

(2) A second tentacle also was spread as the former, and the object was held in between the two tentacles. The animal remained thus for a few seconds and then receded with the tentacles as well as the body expanded.

5. The stimulus was placed at a distance.

(1) One tentacle expanded and touched the object along a side. Other tentacles remained out-spread.

(2) The tentacles bent variously so as to carry the article towards the hypostome.

(3) The object was at length brought in contact with the hypostome.

(4) Immediately after the foreign body touched the hypostome all the tentacles as well as the body contracted

to a certain extent. Finally the Hydra turned back by a vault.

6. The stimulus was brought again almost near the tentacles.

(1) Three tentacles extended and touched the foreign particle on three different sides.

(2) It was pulled almost to the hypostome.

(3) A sudden contraction of tentacles and the body followed.

(4) The Hydra turned back by a vault.

7. The object of stimulus was placed near the tentacles.

(1) Two tentacles expanded and touched the particle and remained so for a few seconds.

(2) All the tentacles and the body gradually moved sideways.

Specimen No. XI.—Period of Sturation—7 days.

The plant tissue was placed at a short distance.

(1) The animal proceeded to it and touched it several times with a tentacle for about a minute.

(2) The Hydra proceeded further and all the tentacles were placed in contact with the object; the animal body gradually contracted and remained attached to it for about $1\frac{1}{2}$ minutes.

(3) Gradual elongation of the body ensued leaving the stimulus at one side still in contact.

(4) The animal again turned towards the foreign particle and coiled its body around it. The object was held in the whorl for over a minute.

(5) The body of the animal became straight and expanded backward, with the particle still in the whorl of the tentacles, retarding from and advancing towards it rhythmically for 2 minutes.

(6) The Hydra receded leaving the object of stimulus at a great distance.

(7) It again advanced with the tentacles expanded so as to hold the object in between the bases of two tentacles bringing it thereby in touch with a side of the hypostome. The specimen remained in this condition for a minute.

(8) A sudden contraction of the body of the Hydra followed and the position of the object was intact. This condition was maintained for one minute.

(9) The Hydra slightly expanded but the object of stimulus was left undisturbed.

(10) The animal body half curled advanced by the side of the object and in touch with it. It was found in this condition for $\frac{1}{2}$ minute and then advanced further leaving the object at a considerable distance.

III

REACTION OF INDIVIDUALS TO A MECHANICAL STIMULUS AFTER THEY HAVE BEEN SUBJECTED TO DARKNESS.

The specimens were subjected to darkness but they were all kept in aquaria maintaining the normal habitat of the Hydra, and their thigmotropic reactions were examined at different periods of their subjection.

The following series of behaviours were the results of application of the same object of stimulus as we used in connection with the foregoing experimentations.

Specimen No. 1.—Subjected to darkness for 1 hour.

1. The mechanical stimulus was applied to a tentacle and the animal exhibited the following reactions.

The animal at once shrank to a globule and all the tentacles contracted to such an extent that they could be identified only as so many dots.

2. A number of tentacles were stimulated at a time and the animal reacted as in No. 1.

3. The stimulus was placed in contact with one tentacle.

The Hydra immediately turned upside down and contracted.

4. The object of stimulus was placed in the whorl of the tentacles. Same reactions as in No. 3 were exhibited.

5. In this case only one tentacle was stimulated.

(1) Immediately after the application of the stimulus all the tentacles began to contract very slowly.

(2) Gradually the animal turned upside-down.

6. Two tentacles were stimulated and the animal receded very slowly.

Specimen No. II.—Subjected to darkness for 2 hours.

1. The stimulus was attached to two tentacles.

One of them remained attached to the object for 3 or 4 seconds and moved away. The other tentacle kept in contact with the object a little longer than the first, say about 6 seconds and also moved away.

2. Only one tentacle was stimulated which remained attached to the object for about 3 seconds and then contracted.

3. The object of stimulus was placed in the whorl of the tentacles.

The animal at once stood up and contracted.

4. Stimulus as in No. 3.

The animal at once became globular.

5. The stimulus was placed on three tentacles and the reaction was received as in No. 4.

6. The stimulus being placed in the whorl the animal at once turned upside-down.

6. The stimulus being placed in the whorl the animal at once turned upside-down.

• *Specimen No. III.—Subjected to darkness for 2 hours.*

1. The stimulus being attached to a tentacle gave rise to the following reactions:—

(1) The object was held in the bends of three tentacles from three different sides, and remained thus for $\frac{1}{2}$ minute.

(2) The object was given a slight pull and then the animal moved away leaving the object.

2. The object was attached to the tips of two tentacles.

(1) The Hydra remained thus for a few seconds, say 5 or 6.

(2) The two tentacles stimulated contracted and the animal moved back.

3. Same as 2.

4. The stimulus was as in No. 2.

The animal at once moved aside.

5. The stimulus was as in No. 2.

The animal at once became globular.

6. Same as in No. 5.

Specimen No. IV.—Subjected to darkness for 3 hours.

The animal being stimulated on one tentacle behaved thus:—

(1) The object was captured by two tentacles.

(2) After $1\frac{1}{2}$ minutes they contracted.

2. Same as No. 1.

But the animal remained in contact with the object for a few seconds only.

3. The stimulus placed in contact with two tentacles evoked the following reactions :—

The tentacle slightly looped the object and then contracted.

4. The stimulus was applied to a tentacle and the reaction was immediate contraction of the whole animal.*

5. Same as No. 4.

6. Same as No 4.

Specimen No. V.—Subjected to darkness for 4 hours.

1. The animal was stimulated on one tentacle only and manifested the following reactions :—

(1) The tentacle remained attached to the object for 1 minute.

(3) Another tentacle proceeded on the object and pulled it from time to time.

(3) Both the tentacles contracted.

The whole process covered a period of 2 minutes.

2. The process of stimulation as in No. 1 was repeated.

(1) The tentacle remained attached to the object for a minute.

(2) It pulled the object and contracted.

3. The stimulus was placed in contact with three tentacles and partly in the whorl.

(1) The animal remained so for 25 seconds.

(2) Before the animal moved aside and contracted the object was once drawn in close to the hypostome.

4. The stimulus was placed on the tips of almost all the tentacles.

The animal at once pushed the object forward and turned back by a vault.

5. The animal was stimulated as before and reacted by contraction of its body to a perfect globule.

6. Same as No. 5.

Specimen No. VI.—Subjected to darkness for 5 hours.

1. We began our experimentation with this specimen by stimulating only one of its tentacles.

(1) This tentacle spread along a side of the object.

(2) Shortly after two other tentacles extended towards the object which was consequently held in the whorl of three tentacles.

(3) The object was thus held for 3 minutes being from time to time gently pulled.

(4) Finally the animal slowly moved aside.

2. Again the stimulus was applied to one tentacle only.

(1) This tentacle extended to a certain distance along a side of the object, remained in contact with it for a few seconds and moved aside.

(2) This tentacle and another touched the object four times in course of one minute, and then the animal moved aside.

3. The same process of stimulation was repeated.

(1) The tentacle which was stimulated slightly curved its tip which remained in contact with the object for 50 seconds.

(2) This tentacle contracted and another proceeded towards the object which was touched by it.

(3) Immediately all the tentacles contracted and the animal turned upside-down.

4. Here too the same process of stimulation was followed.

The tentacle stimulated remained in contact with the object for 20 seconds and the animal moved away.

5. The stimulus was again applied to a tentacle which at once contracted.

6. Same as No. 5.

7. In this case too only one tentacle was stimulated and the result was that the whole animal body contracted and became globular.

8. Here too the stimulus was placed in contact with one tentacle but the animal reacted by contracting all its tentacles.

9. Same as No. 8.

Specimen No. VII.—Subjected to darkness for 6 hours.

1. The stimulus was applied to two tentacles.

(1) Both the tentacles slightly curved and held the object in the bends.

(2) Shortly after the remaining tentacles also extended and touched the object with their curved tips.

(3) The object was held for 2 minutes and 45 seconds, and then the animal with all the tentacles contracted.

2. Only one tentacle was stimulated.

(1) Another tentacle proceeded and the object was held by the two tentacles for over a minute.

(2) One of the tentacles touching the object contracted as well as the body. The other tentacle remained in contact with the object for a few seconds.

(3) Finally the whole animal body with all the tentacles became globular.

3. Here too the same process of stimulation was repeated.

(1) The tentacle stimulated looped the object and made an attempt to pull it.

(2) The object slipped and the animal contracted.

4. The same process was followed again, that is, only one tentacle was stimulated.

(1) The tentacle remained in contact with the object for a few seconds.

(2) Then the animal stood up.

5. In this case also the same process was followed in applying the stimulus.

All the tentacles at once contracted.

6. Same as No. 5.

7. Same as No. 5.

The stimulus was placed in the whorl of the tentacles.
The animal at once stood up and contracted.

Specimen No. VIII.—Subjected to darkness for 7 hours.

1. The stimulus was applied to two tentacles.

(1) The two tentacles remained attached for $2\frac{1}{2}$ minutes pulling the object from time to time.

(2) The whole body contracted.

2. The stimulus as before was placed in contact with two tentacles.

(1) The two tentacles remained thus, that is, in a neutral condition for 40 seconds.

(2) The animal turned upside down and contracted.

3. The stimulus was attached to a tentacle.

(1) Another tentacle proceeded and spread along a side of the object.

(2) The tentacle to which the stimulus was attached remained constantly in contact with the object for $1\frac{1}{2}$ minutes.

(3) The second tentacle moved away and touched the object several times.

(4) Finally the animal contracted.

4. The stimulus was applied to two tentacles.

(1) The two tentacles remained in an extended condition on two sides of the object for 25 seconds.

(2) The animal in an attempt to pull the object towards the hypostome gave it a jerk and moved away.

5. The animal was stimulated as in the preceding case.

(1) The two tentacles remained in contact for 15 seconds.

(2) The object was once pushed forward. The Hydra moved away and slightly contracted.

6. The animal was again stimulated as in the two preceding cases.

The animal at once contracted.

7. The stimulus was placed in contact with the hypostome.

The animal immediately stood up.

8. The same as No. 7.

9. The same as No. 7.

Specimen No. IX.—Subjected to darkness for 8 hours.

1. The stimulus was placed on the tip of three tentacles.

(1) The three tentacles with the object on their tips drew back and the object came nearer the hypostome.

(2) The whole anterior portion of the animal body bent slightly and attempts were made to bring the object still nearer to the hypostome. This process of pulling the object with the help of the three tentacles holding the object and the anterior part of the animal's body continued for 3 minutes.

2. The stimulus was placed in the whorl of the tentacles.

(1) The object was at once looped by the tentacles and remained thus for one minute.

(2) The animal became globular.

3. The stimulus was placed in contact with a number of tentacles.

(1) The object was held in between the tentacles pressing it from different sides. The object was thus held for $2\frac{1}{4}$ minutes.

(2) The animal moved aside.

4. The stimulus was as before.

(1) Three tentacles placed their tips on the object. The object was moved from time to time.

(2) The animal moved aside after $\frac{1}{2}$ minute.

5. The stimulus was applied to one tentacle.

(1) This tentacle touched the object and tried to give it a move.

(2). After $\frac{3}{4}$ of a minute the animal drew back.

(3) The animal at once proceeded to the object, touched it with a tentacle and remained so for $1\frac{1}{2}$ minutes.

(4) The animal moved away. The whole process lasted for $2\frac{1}{4}$ minutes.

6. The stimulus was placed in the whorl.

(1) The object was seized by all the tentacles.

(2) After $\frac{1}{2}$ minute the animal moved aside.

7. The stimulus was placed in contact with a number of tentacles.

(1) They remained in contact for a few seconds.

(2) The animal moved away.

8. Same as No. 7.

9. Same as No. 7.

10. The stimulus was placed in contact with a number of tentacles.

After a few seconds the animal turned back.

11. The stimulus was placed in contact with a number of tentacles.

The animal at once contracted.

12. The stimulus was placed in contact with a number of tentacles.

The animal at once turned back.

13. Same as No. 12.

The animal stood up.

14. Same as No. 12.

The animal raised the anterior portion and contracted.

15. Same as No. 14.

The animal at once contracted itself and became globular.

Specimen No. X.—Subjected to darkness for 36 hours.

1. The stimulus was applied to a few tentacles at a time and the animal exhibited the following reactions :—

(1) Two tentacles extended, each along a side of the stimulating object. Thus holding it, the animal moved back to a certain distance and stopped with the object in the folds of the tentacles for about $\frac{1}{2}$ minute.

(2) The animal contracted with all the tentacles assuming a globular shape.

2. Only one tentacle was stimulated and the following reactions were observed.

(1) Three tentacles extended and held the object in the bends on three different sides.

(2) The object was held for $1\frac{1}{2}$ minutes and then the animal moved away leaving the object.

3. The Hydra was stimulated as before.

(1) The tentacle in contact with the stimulus extended and held the object in the bend.

(2) After 50 seconds the object was drawn towards the hypostome (but not brought in contact with the latter) and remained thus for 40 seconds more.

(3) Contraction of the animal body took place, and consequently, the object was left. Thus the whole process lasted for $1\frac{1}{2}$ minutes.

4. The animal was again stimulated as in the foregoing case.

All the tentacles curled around the object, drew it to the hypostome and moved away from the object immediately.

5. Here too the stimulus was similarly applied and the result was that the tentacle extended and remained in contact with the object only for a few seconds, and then the animal moved away.

6. The animal being similarly stimulated behaved in the following manner.

(1) Same reaction was exhibited as in No. 5; but the period of attachment was prolonged to $1\frac{1}{2}$ minutes.

(2) Then the animal contracted to a globule together with all the tentacles.

7. The same process of stimulation was repeated.

(1) Two tentacles held the object in their angle, one extending below and the other above it.

(2) The object was thus maintained for a minute and then the tentacles slightly contracted and the animal drew back.

8. Only one tentacle was stimulated again.

The animal at once contracted and became globular.

9. The process of stimulation and reaction was as in No. 8.

10. The stimulus was given as before.

(1) The Hydra at once became globular and came aside from the object.

(2) It extended its tentacles and assumed the position upside down.

11. Here too only one tentacle of the Hydra was stimulated.

(1) The body of the Hydra contracted first.

(2) Then the tentacles also contracted and the animal drew back.

12. The process of stimulation was again repeated and the animal moved away from the object.

13. The stimulus was placed in the whorl of the tentacles.

(1) The Hydra waved the tentacles sideways for 12 seconds.

(2) A sudden contraction of the tentacles only took place.

Specimen No. XI.—Subjected to darkness for 58 hours.

1. The stimulus was applied to two tentacles.

(1) One of the tentacles curved round the object and the other remained straight.

(2) The curved tentacle was moved away from the object whereas the second tentacle remained as before for a few seconds and then the object was pushed aside.

(3) By movements of the animal body the object happened to come in contact with the tentacles. The tentacles were so arranged that the object was taken in their whorl and was gradually carried on to the hypostome.

(4) Immediately the object was pushed forward and two of the tentacles remained in association with it for about $1\frac{1}{2}$ minutes.

(5) The animal slowly receded.

2. The object was placed at a short distance from the tentacles.

(1) One tentacle curled around the object.

(2) Two other tentacles also were extended and held the object from three different sides.

(3) After a few seconds the animal drew back slowly.

3. The stimulus was applied to one of the tentacles.

(1) Two tentacles were extended, one above the object and the other below it.

(2) The object was left by a sudden contraction of the tentacles.

4. The animal was stimulated as in the preceding case and the animal suddenly contracted and left the object.

5. Here too only one tentacle was stimulated.

(1) The object was taken in the whorl of three tentacles and was held for about 50 seconds.

(2) The Hydra moved aside.

6. The stimulus was placed again in contact with one tentacle.

(1) The animal moved towards the object which was then held in the whorl of the tentacles.

• (2) Gradually the hypostome touched the object but immediately it was taken aside.

(3) The object was held in between two tentacles for a few seconds and then the animal moved aside.

7. The process of stimulation was unaltered.

(1) All the tentacles immediately contracted.

(2) The anterior portion of the animal body moved away from the object.

8. The stimulus was applied to two tentacles.

(1) A third one extended and touched the object.

(2) The above three tentacles contracted and the animal moved away from the object.

9. The stimulus was applied as in No. 8.

Immediate contraction of the two tentacles took place.

10. The stimulus was placed in the whorl of the tentacles.

At once the animal became globular.

11. The stimulus was placed near two tentacles.

As soon as the tentacles touched the object they contracted and also the animal moved aside.

Specimen No. XII.--Subjected to darkness for 82 hours.

1. The stimulus was applied to one tentacle.

(1) The tentacle expanded around the object and remained in contact with it for one minute moving it to and fro from time to time.

(2) The tentacle was drawn aside but the animal remained stationary.

2. The stimulus was applied to three tentacles.

(1) All three tentacles curved around the object from three different sides.

(2) They remained in contact with the object for 2 minutes moving it to and fro.

(3) The animal left the object and slowly moved away.

3. The stimulus was applied to two tentacles.

(1) The two tentacles expanded on the object and remained as such for $2\frac{1}{4}$ minutes.

(2) The animal stood up.

4. The stimulus was given to a tentacle.

(1) The tentacle was first slightly raised and expanded on the object.

(2) The object was held in this way for a few seconds.

(3) The animal raised the hypostome and moved aside.

5. The stimulus was applied to two tentacles.

(1) The object was held in the bends of the two tentacles.

(2) Both the tentacles remained in contact with the object for 45 seconds pulling the object towards the hypostome and pushing it off several times; and then the animal moved away.

3. This time only one tentacle was stimulated.

The tentacle at once repelled the object.

Specimen No. XIII.—Subject to darkness for 105 hours.

1. The stimulus was applied to one tentacle only.

(1) Four tentacles extended and their tips curled around the object from four different sides.

(2) The object was thus held for $1\frac{1}{2}$ minutes and the animal relaxed its hold on the object by contraction of the tentacles.

2. The stimulus was placed in the whorl of three tentacles.

(1) These three tentacles caught the object and pulled it towards the hypostome.

(2) The object was pushed off as soon as it came in contact with the hypostome.

(3) The animal again proceeded to the object and touched it by two tentacles. It remained as such for a few seconds and then became globular.

3. One of the tentacles was stimulated and the animal behaved in the following manner.

(1) Another tentacle beside the one stimulated extended and both of them curved around the object.

(2) The object was thus held for $1\frac{1}{2}$ minutes and was from time to time moved forward and backward by the tentacles.

(3) Then the animal left the object and moved away.

4. The object of stimulus was placed in touch with two tentacles.

(1) Both the tentacles expanded, one on the object and the other below it.

(2) The object was thus held for a few seconds and the animal made an attempt to turn the object sideways.

(3) Suddenly the anterior part of the animal body with all the tentacles contracted.

5. The stimulus was placed very near the tentacles and the following reactions were observed.

(1) The six tentacles expanded all at a time and held the object from different sides.

(2) The object was thus held for 2 minutes and 50 seconds and then the animal moved aside.

6. The animal being stimulated on one of its tentacles behaved as follows.

(1) The tentacle curved round the object which was drawn close to the hypostome.

(2) The animal drew aside.

7. The stimulus was placed very near a tentacle and gave rise to the following reactions :—

(1) The tentacle expanded and touched the object.

(2) Immediately the animal drew away.

8. The stimulus was placed in touch with a tentacle.

About 4 or 5 tentacles protruded and touched the object. The animal moved away at once.

9. The stimulus was applied as in No. 8.

The animal at once contracted its tentacles and drew back.

10. Same as No. 9.

11. The animal was again stimulated as in No. 8.

The animal at once contracted its tentacles, raised its head and expanded in a different direction away from the object.

12. In this case too the animal was similarly stimulated.

The animal at once contracted its tentacles with the anterior portion of the body.

13. Same as No. 9.

Specimen No. XIV.—Subjected to darkness for 153 hours.

1. The stimulus was applied to one of the tentacles and the following reactions were observed.

(1) Another tentacle proceeded and the object was held between the two tentacles for $2\frac{1}{4}$ minutes.

(2) The animal slightly raised its anterior portion and moved aside.

2. In this case the stimulus was applied to two tentacles.

(1) Another tentacle proceeded and the object was held by three tentacles, two being on the object and the other below it.

(2) An attempt was made to draw the object towards the hypostome when it appeared that the grasp became slackened and the object escaped.

(3) Then the animal body became globular.

3. The object was placed in contact with a tentacle.

(1) Other tentacles too proceeded towards the object.

(2) As soon as the tentacles touched the object the animal ran away with it to a certain distance.

(3) Then the object was left and the animal body with the tentacles slightly contracted.

4. The stimulus was applied to one of the tentacles.

(1) The object was pulled on the hypostome by the same tentacle alone.

(2) The body contracted with all the tentacles expanded and the object was thus held for about a minute.

(3) The animal body elongated, and consequently, the object was left.

5. The stimulus was applied to two tentacles.

(1) The object was slightly looped by the two tentacles.

(2) The animal left the object immediately and passed away.

6. One of the tentacles was stimulated.

(1) The tentacle was slowly raised from the object.

(2) The oral end of the animal body was bent aside.

7. The animal was stimulated as in No. 6.

The animal at once became globular.

8. Same as No. 7.

9. The animal was stimulated as in No. 6.

It reacted by moving away from the object.

10. The animal being again stimulated as in the foregoing case first reacted by a sideway movement and contracted.

IV

GENERALISATION AND CONCLUSION

In consideration of the nature of movements of the animal body as a whole (with tentacles) and the period of its association with the object of stimulus we will arrange the foregoing behaviours of the Hydra in the following manner:—

1	2	3	4	5	6	7
General condition of the specimen.	Serial No. of the specimen.	Special feature of the specimen.	Serial No. of the behaviours.	Nature of reactions.	Period required for the whole set of behaviours of the specimen.	REMARKS.
Normal.	I	1..... 2..... 3..... 4..... 5..... 6..... 7..... 8..... 9..... 10..... 11..... 12..... 13.....	Negative Negative Negative Negative Negative Negative ¹ Negative Negative Negative Negative Negative Negative Negative		¹ The object was held for 20 seconds in the bend of one of the tentacles before the Hydra was finally oriented. This short attachment may be considered to have been obligatory under the circumstances and in view of the subsequent reactions may not be taken as a sign of the Hydra's positive reaction.

II	14.....	Negative	This was a case of trial and error.
	15.....	Negative	
	16.....	Negative	
	17.....	Negative	
	18.....	Negative	
	19.....	Positive *	
	1.....	Negative	
	2.....	Negative	
	3.....	Negative	
	4.....	Negative	
	5.....	Negative	
	6.....	Negative	
	7.....	Negative	
	8.....	Negative	
	9.....	Negative	
	10.....	Negative	
	11.....	Negative	
	12.....	Negative	
	13.....	Negative	
	14.....	Negative	
	15.....	Positive	
III	16.....	Negative	This was a case of trial and error.
	17.....	Positive	
	18.....	Positive	
	19.....	Positive	
	20.....	Positive	
	1.....	Negative	
	2.....	Negative	
	3.....	Negative	
	4.....	Negative	
	5.....	Negative	
	6.....	Negative	
	7.....	Negative	
	8.....	Positive *	
	9.....	Negative	
	10.....	Negative	

1	2	3	4	5	6	7
General condition of the specimen.	Serial No. of the specimen.	Special feature of the specimen.	Serial No. of the behaviours.	Nature of reactions.	Period required for the whole set of behaviours of the specimen.	REMARKS.
Normal— <i>concl'd.</i>	III — <i>concl'd.</i>	11..... 12..... 13..... 14..... 15..... 16..... 17..... 18..... 19..... 20.....	Negative Negative Negative Positive Positive Positive Positive Positive		
Subjected to starvation.		Starved for three days.	1..... 2..... 3.....	Positive Positive Positive		¹ Reaction incomplete.
	II	Starved for three days.	1..... 2..... 3..... 4..... 5..... 6..... 7..... 8.....	Positive Positive Positive Positive Positive Positive Negative ² Positive		² In the preceding and the subsequent cases of positive reaction the tentacles of the Hydra were stimulated. Therefore it may safely be conjectured that the present negative reaction of the Hydra may have been due to the stimulus applied to its body.
	III	Starved for three days.	1..... 2.....	Positive ³ (mostly) Positive		³ This was a case of trial and error.

* This behaviour was so very complicated that it evolved a series of reactions.

IV	Starved for four days.	3..... 4..... 5.....	Negative Positive Positive
V	Starved for four days.	1..... 1..... 2..... 3..... 4..... 5.....	Positive* Positive Positive Positive Positive
VI	Starved for four days.	1..... 2.....	Positive Positive
VII	Starved for five days.	1..... 2..... 3..... 4..... 5.....	Positive Positive Positive Positive Positive
VIII	Starved for five days.	1..... 2..... 3..... 4..... 5.....	Positive Positive Negative Negative (?) Positive
IX	Starved for six days.	1..... 2.....	Positive Negative (?)
X	Starved for six days.	1..... 2..... 3..... 4..... 5..... 6..... 7.....	Positive Positive Positive Positive Positive Positive Positive

1	2	3	4	5	6	7
General condition of the specimen.	Serial No. of the specimen.	Special feature of the specimen.	Serial No. of the behaviours.	Nature of reactions.	Period required for the whole set of behaviours of the specimen.	REMARKS.
Subjected to starvation.— <i>concl'd.</i> Subjected to darkness.	XI	Starved for seven days.		Positive ¹		¹ This behaviour was so very complicated that it evolved a series of reactions.
	I	Subjected to darkness for one hour.	1..... 2..... 3..... 4..... 5..... 6.....	Negative Negative Negative Negative Negative Negative		
	II	Subjected to darkness for 2 hours.	1..... 2..... 3..... 4..... 5..... 6.....	Positive ¹ Positive ¹ Negative Negative Negative Negative	2.13 P.M. to 2.35 P.M. (22 minutes).	¹ The period of association was very short. Negative reactions began at 2.25 P.M. Therefore the time required for the revival of normal condition was 12 minutes.
	III	Subjected to darkness for 2 hours.	1..... 2..... 3..... 4..... 5..... 6.....	Positive Positive ¹ Positive ¹ Negative Negative Negative		¹ The period of association was very short. Negative reaction commenced at 2.50 P.M. Therefore the time required for the revival of the normal stage was 15 minutes.

IV	Subjected to darkness for 3 hours.	1..... 2..... 3..... 4..... 5..... 6.....	Positive Positive Positive Negative Negative Negative	3.13 P.M. to 3.39 P.M. (26 minutes).	Negative reactions commenced at 3.26 P.M. Therefore the time required for restoration is 13 minutes.
V	Subjected to darkness for 4 hours.	1..... 2..... 3..... 4..... 5..... 6.....	Positive Positive Positive Negative Negative Negative	11 A.M. to 11.37 A.M. (37 minutes).	Negative reactions commenced at 11.25 A.M. Therefore the time required for restora- tion is 12 minutes. The mean of the restorative periods of II to V is 13 minutes.
VI	Subjected to darkness for 5 hours.	1..... 2..... 3..... 4..... 5..... 6..... 7..... 8..... 9.....	Positive Positive Positive Positive Negative Negative Negative Negative Negative	12 A.M. to 12.50 P.M. (50 minutes).	Negative reactions began to appear at 12.25 P.M. Hence, the time required for restoration is 25 minutes.
VII	Subjected to darkness for 6 hours.	1..... 2..... 3..... 4..... 5..... 6..... 7..... 8.....	Positive Positive Positive Positive Negative Negative Negative Negative	12.15 P.M. to 12.55 P.M. (40 minutes).	Negative reactions began at 12.45 P.M. Therefore the restorative period is 30 minutes. The mean of the restorative periods of VI & VII is 27.5 minutes.

The mean has been taken as the difference in value of the members is very small.

1	2	3	4	5	6	7
General condition of the specimen.	Serial No. of the specimen.	Special feature of the specimen.	Serial No. of the behaviours.	Nature of reactions.	Period required for the whole set of behaviours of the specimen.	REMARKS.
Subject to darkness.— <i>contd.</i>	VIII	Subjected to darkness for 7 hours.	1..... 2..... 3..... 4..... 5..... 6..... 7..... 8..... 9.....	Positive Positive Positive Positive Negative Negative Negative Negative	2.20 P.M. to 3 P.M. (40 minutes).	Negative reactions began at 2.50 P.M. Hence, the time required for restoration is 30 minutes. The mean of the restorative periods of ; VI to VIII is 28.3 minutes.
	IX	Subjected to darkness for 8 hours.	1..... 2..... 3..... 4..... 5..... 6..... 7..... 8..... 9..... 10..... 11..... 12..... 13..... 14..... 15.....	Positive Positive Positive Positive Positive Positive Positive Positive Positive Positive Negative Negative Negative Negative Negative	3 P.M. to 3.55 P.M. (55 minutes).	Negative reactions commenced at 3.44 P.M. Therefore the time required for the revival of normal condition was 44 minutes.

X	Subjected to darkness for 36 hours.	1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13.	Positive Positive Positive Positive Positive Positive Negative Negative Negative Negative Negative Negative	3.15 P.M. to 4.30 P.M. (75 minutes).	Commencement of negative reactions was noticed at 4 P.M. Therefore the effect of darkness that reversed the negative thigmotropism to positive one was retained for 45 minutes, after which the animal was restored.
XI	Subjected to darkness for 58 hours.	1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11.	Positive Positive Positive Positive Positive Negative Negative Negative Negative Negative	2 P.M. to 3.15 P.M. (75 minutes).	Negative reactions commenced at 2.50 P.M. Therefore the time required for the revival of normal thigmotropism was 45 minutes.

The mean has been taken as the difference in value of the members is very small.

1	2	3	4	5	6	7
General condition of the specimen.	Serial No. of the specimen.	Special feature of the specimen.	Serial No. of the behaviours.	Nature of reactions.	Period required for the whole set of behaviours of the specimen.	REMARKS.
Subject to darkness. —contd.	XII	Subjected to darkness for 82 hours.	1.....	Positive	1.55 P.M. to 2.50 P.M. (55 minutes).	Negative reactions commenced at 2.40 P.M. Therefore the time required for the revival of normal thigmotropism was 45 minutes.
			2.....	Positive		
			3.....	Positive		
			4.....	Positive		
			5.....	Positive		
			6.....	Positive		
			7.....	Negative		
			8.....	Negative		
			9.....	Negative		
			10.....	Negative		
			11.....	Negative		
			12.....	Negative		
	XIII	Subjected to darkness for 105 hours.	1.....	Positive	1 P.M. to 2.10 P.M. (70 minutes).	The beginning of negative reactions was noticed at 1.45 P.M. Therefore the time required for the revival of normal thigmotropism was 45 minutes.
			2.....	Positive		
			3.....	Positive		
			4.....	Positive		
			5.....	Positive		
			6.....	Positive		
			7.....	Positive		
			8.....	Positive		
			9.....	Negative		
			10.....	Negative		
			11.....	Negative		
			12.....	Negative		
			13.....	Negative		

If we consider case 6 as transitional, the true negative reactions which commenced at 4.30 P.M. may be counted from No. 7.

Therefore the time required for the restoration of normal thigmotropism was 45 minutes.

The mean of the restorative periods of IX to XIV is 45.6 minutes.

3.45 P.M. to
4.40 P.M.
(55 minutes.)

Positive
Positive
Positive
Positive
Positive?
Negative
Negative
Negative
Negative

1.....
2.....
3.....
4.....
5.....
6.....
7.....
8.....
9.....
10.....

Subjected to darkness
for 158 hours.

XIV

The mean has been taken as the difference in value of the members is very small.

In view of the above abridgment we may draw the following conclusions regarding the behaviours of the Hydra.

(1) All the reactions of the Hydra to a mechanical stimulus which is not its food under normal conditions are negative; but such behaviours of one and the same individual are transformed into positive ones after continued repetitions of the stimulus; that is, the negative reactions turn to be a food-taking reaction.

(2) When the Hydra is subjected to starvation for a certain period, reversal of its thigmotropism occurs. That is to say, the Hydra at once behaves in such a way as manifests its food-taking reactions towards the mechanical stimulus which under normal circumstances is violently rejected.

(3) The reversing influence of starvation increases along with its prolongation.

(4) The behaviours of the Hydra, when it is subjected to the influence of darkness, evolve food-taking reactions to a mechanical stimulus. From this it follows, then, that the state of the Hydra concomitant to its subjection to the influence of darkness is equivalent to its hungry condition.

The remark column of the abridgment shows that the subjection of the animal to darkness for one hour has no effect whatsoever upon the normal thigmotaxis of the Hydra, and in consequence, it behaved negatively towards the object of stimulus. There was a marked change of behaviours on the part of the Hydra after it was influenced by darkness for two hours. But then, three hours' subjection to darkness (that is, increment of the latter by one hour) did not produce any additional change in the behaviours of the specimen. Prolongation of the period of subjection by another hour did not produce any change of behaviour either.

Now it is noticed, curiously enough, that the behaviour of the Hydra was almost doubly changed due to its subjection to darkness for five hours. That is, the restorative period was prolonged from 13 minutes to 25 minutes. Practically there is very little change in the three subsequent readings the mean of the restorative periods of which is 28·3 minutes. At the end of the eighth hour of subjection the influence of darkness was so intense that the restorative period was raised from 28·3 minutes to 45·6 minutes.

(5) So, by re-arranging the mean readings in sequence we obtain a definite relation between the length of subjection to darkness of the Hydra and the reversal of its thigmotropism as well as the time required for restoration of its reversed thigmotaxis to a normal stage. The Hydra's thigmotropism is first reversed after it is influenced upon by darkness for two hours. The next three hours' subjection doubles (practically) the period of restoration. But the result of the next three hours of subjection instead of doubling the restorative period increases it by 1·5. Then the figure remains constant. To be more definite, we may state that any addition to the length of the Hydra's subjection to darkness after the 8th hour does not vary its restorative period, that is, the animal henceforward will react positively to a mechanical stimulus ceaselessly for 45 minutes.

To give this statement the form of a definite law, an intense study of the subject with a view to obtain more precise results is necessary; and we conjecture that some day in near future it may not be impossible to declare that Weber's "Law of liminal intensity and liminal difference" holds good also in the case of the Hydra's sensibility.

(6) The tenacity of Hydra's life is wonderful. In connection with the present thigmotropic studies mention has been made of the animal's starvation for seven days

only; but we observed a specimen that kept without food as long as twenty-one days. The creature became awfully lean before it succumbed.

(7) The Hydra is able to perform various movements of its body. In course of the aforesaid behaviours it involved the following different kinds of movement :— •

- (1) Flight.
- (2) Waving of the tentacles.
- (3) Partial contraction of the tentacles.
- (4) Contraction of the tentacles to so many dots.
- (5) Expansion of the tentacles.
- (6) Partial contraction of the animal body.
- (7) Contraction of the animal body to a globule.
- (8) Expansion of the body.
- (9) Performance of a vault.
- (10) Bending of the body.
- (11) Pushing off of an object.
- (12) Assumption of an upright posture.
- (13) Turning upside-down.
- (14) Waving of the anterior portion of the body with the foot fixed.
- (15) Making loops of the tentacles.

In how far an animal is able to preceive the external objects largely depends upon what it is able to do with the objects. This capability involves a variety of movements each of which has some adaptive value and is contributive towards the mental development of the animal.

BOTANY

COMMENTATIONES PHYTOMORPHOLOGICAE
ET PHYTOPHYSIOLOGICAE

III

EICHHORNIA STUDIES

BY

PAUL BRÜHL, D.Sc., AND ATULCHANDRA DUTTA, M.Sc.

ON THE DISTRIBUTION OF POTASSIUM IN
EICHHORNIA SPECIOSA

As already known, *Eichhornia speciosa* is one of the plants which absorb considerable amounts of potassium from the medium in which they grow. It is therefore of some interest to have more precise information concerning the distribution of potassium within the different parts of the plant. Such knowledge might supply indications as to the most profitable utilisation of the potash contents. According to WEEVERS (*Recueil des Travaux botaniques Néerlandais*, Vol. VIII, p. 289) potassium accumulates in plants, particularly in reserve organs and in the growing-point, in medullary rays and in the cambium; more is found in the phloem than in the xylem, and it occurs markedly in the unligified parts of the cortical tissue. WEEVERS states that it is concentrated in the vacuoles, but is not present in the chromatophores. WILLSTATTER and STOLL's researches have shown that neither Chlorophyll a nor Chlorophyll b contain any potassium, magnesium being the only metallic constituent of these compounds.

Professor MOLISCH (*Mikrochemie der Pflanze*, third edition, 1923) recommends alcoholic solution of platinic chloride and a solution of sodium cobaltinitrite as reagents for the determination of the presence of potassium in the tissue-elements of plants. MACALLUM (in *Journal of Physiology*, Vol. XXXII, p. 95)^{*} recommends the following method of preparing the reagent—Dissolve 20 grams of cobalt nitrite and 35 grams of sodium nitrite in 75 cc. of dilute acetic acid (10 cc. glacial acetic acid diluted with water to 75 cc.). As soon as the evolution of nitrous fumes has ceased, dilute with water to 100 cc. Filter off any precipitate formed due to the presence of potassium in the reagents. According to MOLISCH the reagent prepared as described above loses its efficacy after one or two days and has to be freshly prepared after the lapse of a couple of days, which is certainly a drawback.

F. CLEYREFET (in *Bull. Soc. Chim. Belg.*, 1922, XXXI, pp. 417-420, see *Abstracts, Chem. Soc., London*, No. 725, March, 1923, II, 181) prepares two solutions: (1) Dissolve 28.6 gr. of cobalt nitrate in 50 cc. of glacial acetic acid and dilute with water to 500 cc.; (2) Dissolve 180 gr. of sodium nitrite in 500 cc. of water. For use add (1) slowly to (2); let stand for 24 hours and filter, if necessary.

POZZI-ESCOTT (in *Bull. Soc. Chim. Belg.*, 1923, XXII, p. 227, quoted in *Journ. Chem. Soc., London*, No. 729, July, 1923) points out that De Koninck published a similar process some years ago and states that the procedure is followed officially in the United States. DE KONINCK dissolves 5 parts of cobalt chloride or cobalt nitrate and 10 parts of sodium nitrite in 100 parts of water and adds some drops of acetic acid.

The process described by CLEYREFET has the advantage that the two stock solutions can be kept;

suitable quantities of the reagent can then be prepared for testing plant tissues by mixing equal volumes of (1) and (2).

It appears, however, that the solution recommended by TREADWELL (*Treadwell-Hall, Analytical Chemistry, Vol. I, p. 50*) keeps quite well for a long time in the temperature conditions obtaining in the botanical laboratory at Baliganj. TREADWELL, who quotes WILLIAM J. BRAY, recommends to dissolve 100 gms. sodium nitrite in 200 cc. of water, adding 60 cc. of 30 per cent. of acetic acid and 10 gms. of hexahydrated cobalt nitrate; after standing for at least 24 hours, the solution is filtered and diluted to 400 cc.

In order to test the permanence of the reagent prepared according to Treadwell with reference to its applicability to the microchemical proof of the presence of potassium in the tissues of plants, we compared the results of tests made by using a solution of Treadwell's reagent prepared in May, 1921, with those obtained by a freshly prepared solution as well as by a solution prepared according to the directions given in Cleyrefet's paper. We have not found any difference when using consecutive sections through different parts of *Eichhornia speciosa*. The reagent prepared according to Treadwell's recipe appears therefore to deserve to be recommended for locating potassium in various parts of plant tissues.

It will appear from the following detailed account that the size of the crystals formed on the treatment of sections with sodium cobaltinitrite varies from less than one μ to more than six μ . Examined between crossed nicols they prove themselves to belong to the cubical system. They are isodiametrical and are bounded by at least twelve faces, which form sets of pairs of parallel faces. Molisch declares the crystals to be pentagonal dodecahedra. This agrees with the characters just

described. It is however noticeable that the faces of the larger crystals, such as shown in fig. 4 on plate I, are more or less irregular hexagons, whilst in other cases the face-view of the crystals is square or oblong. This suggests such combinations as

$$\{100\} \{210\} \text{ and } \{111\} \{201\}.$$

The crystals are deep orange-yellow; the yellow colour is visible even at high magnifications. In the figures only the crystals of potassium cobaltinitrite are shown; other granular contents of the tissue elements—starch, chlorophyll grains and others—are omitted.

The following is a detailed statement concerning the distribution of potassium in the different parts of the Water Hyacinth.

1. **THE RUNNER.**—The crystals of potassium cobaltinitrite in the epidermis and the subepidermal layers are pretty numerous, but less than 1μ in diameter. As we proceed towards the centre the crystals increase in size from 1 to 6μ , whilst in the central parts crystals from 6μ to 14μ in diameter are mixed with smaller crystals. In the bast fibres crystals are pretty numerous.

2. **THE PETIOLE.**—In the epidermis numerous small crystals make their appearance on applying the reagent. The same holds good with regard to the subepidermal layers. In the parenchymatous bundle-sheath the crystals formed are minute and numerous. In the cells of the diaphragms the crystals are uniformly distributed and numerous, and in the cells constituting the longitudinal walls of the air-canals the plentiful crystals have the form of minute grains.

3. **THE PSEUDOBULB.**—In the more centrally situated cells of the diaphragms crystals are more numerous than in the peripheral ones; in either case they are very minute. In the other cells of the pseudobulb the size

and amount of the crystals is similar to what is observable in bulbless petioles.

4. **THE LEAF BLADE.**—The crystals found in the epidermis of the upper leaf-surface are regularly more numerous but much smaller— 2μ and less—than in the cells of the lower epidermis where the larger crystals, which reach 3.5μ in diameter, prevail. A large number of crystals are found in the parenchymatous bundle-sheath and the surrounding parenchyma. The palisade and spongy tissue cells contain only a small number of crystals, and the crystals observable in the diaphragms are very scanty.

5. **THE STEM.**—The crystals, which are usually 2μ to 6μ in diameter, although larger and smaller crystals occur also, are almost uniformly distributed over all the cells of a cross-section, but it appears that they are more numerous in the cells surrounding each vascular bundle. The quantity of potassium indicated by the crystals of potassium cobaltinitrite is greater in the upper than in the lower part of the stem and evidently reaches there its maximum.

THE ROOTLETS.—Crystals are most numerous in the ground-tissue, the number diminishing from the centre outwards; they are nearly uniform in size and less than 1μ in diameter.

THE AXIS OF THE INFLORESCENCE.—Numerous crystals of Potassium cobaltinitrite are found in all the cells and they exceed in number those observed in the cells of the leaf-stalk; they vary from 1μ to 3.5μ in diameter.

THE OVARY.—Crystals are numerous both in the cells of the wall of the ovary, and in those of the ovules.

BUDS.—Longitudinal sections exhibit numerous crystals.

THE LIGULAR SHEATH.—Crystals are found in all the cells, but they are most numerous in the bundle-sheaths.

We may add that the numerous starch-grains found in the cells of the stem during the cold season gradually disappear as the hot season advances.

From the foregoing account it will be seen that potassium occurs in all parts of the Water Hyacinth, although in somewhat varying quantities, the part richest in it being evidently the stem (often wrongly called the root).

BOTANICAL LABORATORY,
UNIVERSITY COLLEGE OF SCIENCE,

The 1st of September, 1923.

Explanation of Figure.

PLATE I.

Fig. 1. Transverse section through the runner : peripheral portion, $\times 60$.

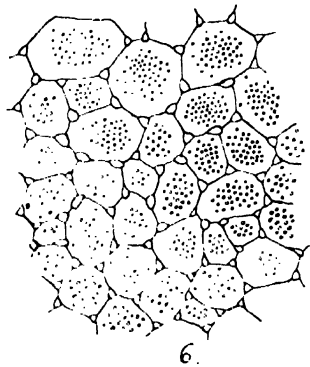
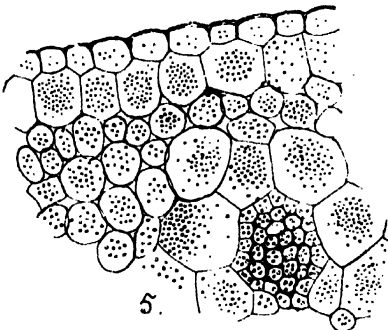
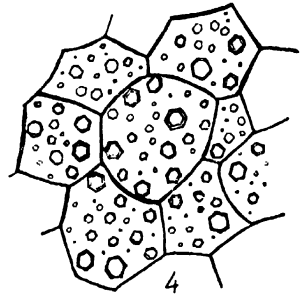
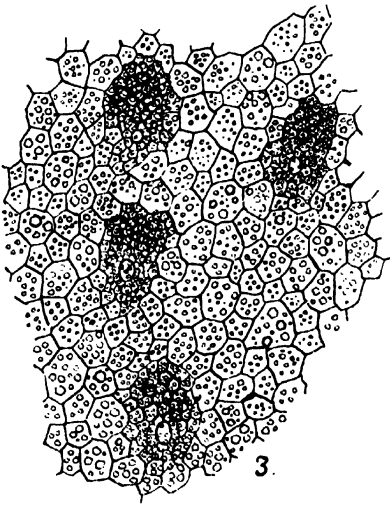
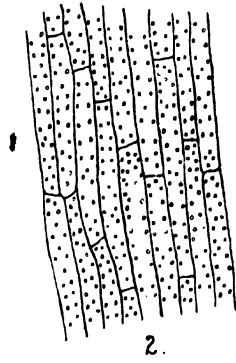
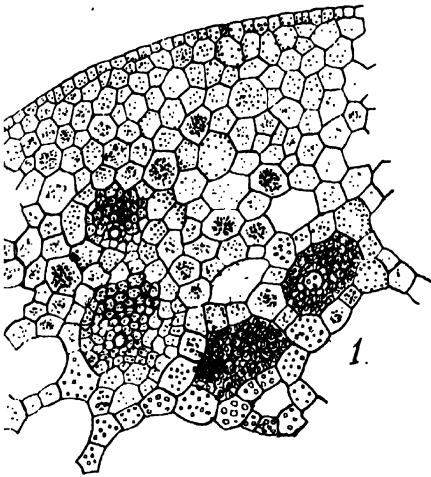
Fig. 2. Face-view of the epidermis of the runner, $\times 120$.

Fig. 3. Transverse section through the central part of the runner, $\times 60$.

Fig. 4. Some cells of the central part of the runner, $\times 120$.

Fig. 5. Transverse section through the petiole : peripheral part, $\times 120$.

Fig. 6. Diaphragm from the petiole, $\times 120$.



Explanation of Figure.

PLATE II.

Fig. 7. Transverse section of a rootlet : pith below, innermost layers of cortex uppermost, $\times 310$.

Fig. 8. Part of epidermis of the upper surface of the leaf, $\times 120$.

Fig. 9. Part of the epidermis of the lower leaf-surface, $\times 120$.

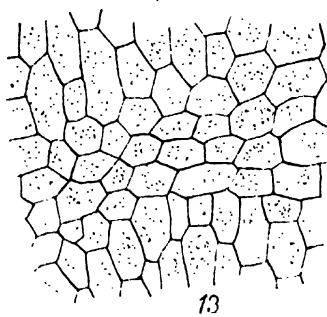
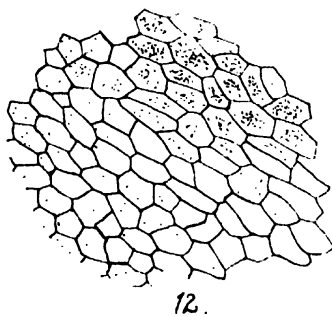
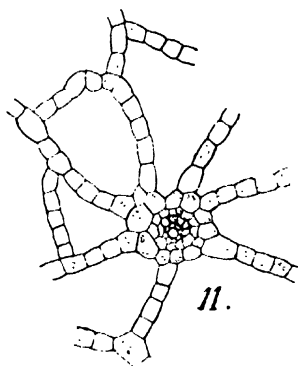
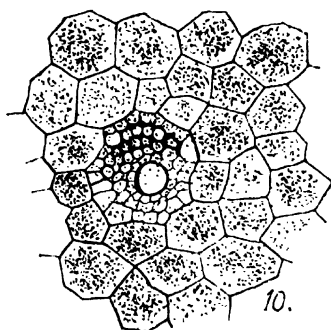
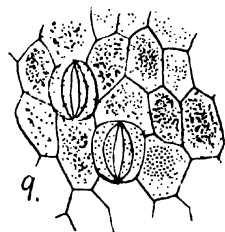
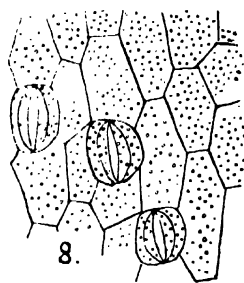
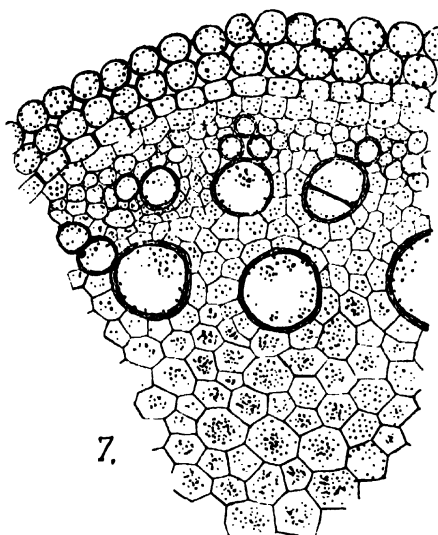
Fig. 10. Part of a cross section through the stem showing a fibro-vascular bundle, $\times 100$.

Fig. 11. Part of a cross-section through the central region of the pseudo-bulb.

Fig. 12. Diaphragm from the central part of the pseudo-bulb, $\times 100$.

Fig. 13. Part of the longitudinal wall of an air-canal, $\times 100$.

PLATE II



Drawn by A. C. Dutta

CHEMISTRY

THE PRECIPITATION OF SUSPENSIDS BY ELECTROLYTES

BY

PROFESSOR JNANENDRANATH MUKHERJEE, D.Sc. (LOND.)

INTRODUCTION

Investigations on the precipitation of suspensoids have so far been restricted to the influence that the nature and the concentration of an electrolyte have on the process. There is, however, no exact investigation of other factors which determine the rate of precipitation. One finds in the literature quite contradictory observations. Thus Woudstra (Z. Phys. Chem. 1908, 61, 607) found that the greater the concentration of the colloid the smaller the concentration of an electrolyte necessary to precipitate it. This is the reverse of what Freundlich (Z. Phys. Chem. 1903, 44, 129) has observed with arsenious sulphide sols. The need for a detailed investigation will also be evident from the fact that even the order of the coagulating power of various electrolytes, as observed by different investigators for the same colloid, is not the same (*cf.* Reprint No. 2, Table 1). Besides it is not known what influences the degree of dispersion, the concentration of the peptising substance, and temperature, have on the process. For example, Linder and Picton (T., 1905, 87, 1906) remark that the precipitating concentration of a salt is generally diminished by rise of temperature. But Paine (Proc. Camb. Phil. Soc., 1912, 16, 430) observes that there is no very marked change in the rate of coagulation on raising the temperature to 100°C. It is not *a priori* possible to say whether the differences between these observations are due to the fact that the colloidal systems studied are different; for there is no data to show that these influences have been at all studied.

Recently Smoluchowski (Zeit. Phys. Chem., 1917, 92, 129) has advanced a mathematical theory regarding the rate and manner in which the coalescence of the particles proceed with time.

Ultramicroscopic investigations have been carried out by Zsigmondy (Z. Phys. Chem. 1918, 92, 600), Westgren and Reitstötter (*ibid.*, 1918, 92, 750) and Westgren (Arkiv. Kem. Min. Geol., 1918, 7, No. 6) and Kruyt and Arkel (Rec. Travaux Chim. Pays-Bas ; 1920, 39, [4], P. 656; 1921, 40, [4], 169). Though there is in some cases tolerable agreement between the theoretical and observed values, the agreement is on the whole unsatisfactory. The limitations of the ultramicroscope make it desirable to have an independent test of the equations deduced by Smoluchowski. The interest of this work is that on the basis of this theory it is possible to compare quantitatively the influence of various factors on the rate of precipitation. It is also possible to determine and compare the percentage of the total encounters between the particles of the colloid, which leads to a stable coalescence of the colliding particles.

The importance of the charge of the colloidal particles on their stability has been well established through the works of Hardy (J. of Physiol. 1899, 24, 288), Burton (Phil. Mag. [6], 1906, 12, 472; 1909, 17, 583), Galecki (Z. Anorg. Chem., 1912, 74, 179), Ellis (Z. Phys. Chem., 1914, 89, 145), and Powis (*ibid.*, 91, 186). The manner in which the rate of precipitation depends on the potential of the double layer surrounding the particles requires a thorough investigation. Only from a knowledge of the quantitative relationship existing between the percentage of successful collisions and the potential of the double layer, a definite theory of the protecting influence of the electric charge can be expected. The usual methods of determining the electric charge are of two kinds, namely :

- (a) the microscopic method of measuring the movement of a particle in different layers ; and
- (b) the macroscopic method of measuring the movement of a colloid-electrolyte (and water) boundary.

Of these the first method is not suitable for fine particles showing great Brownian movement for several reasons. The second method is usually employed and was tried by the writer. The method was found to be quite unsuitable for experiments with arsenious sulphide and this led to an investigation as to the sources of error. It is generally known that the limits of error by the usual methods are as great as 20%. An improved method has been developed but unfortunately the time taken up by this investigation was too great to permit of

a determination of the variation in the percentage of successful collisions with the potential of the double layer.

For an examination of the influence of the degree of dispersion and of the colloid content, sols of arsenious sulphide are particularly suitable. The concentration of the colloid and the size of the particles can be widely varied with ease. The sol can be prepared easily and is of great stability. This sol has the additional interest that observations on this system have long been the basis of theoretical treatment of the subject.

For the work directed to test Smoluchowski's theory the fine gold sols obtained by the nucleus method of Zsigmondy are particularly suitable. They satisfy to a large extent the theoretical condition of uniformity in the size of the particles.

The subjects dealt with in the present thesis are given below.

I. Experimental.

Part I.—Experimental procedure for arsenious sulphide sols with a few remarks on the suitability of different methods.

Part II.—Influence of the dilution of the sol on the rate of precipitation of metal sulphide sols.

Part III.—Influence of the quality of the sol, that is, of its degree of dispersion and of its colloid content: arsenious sulphide sols.

Part IV.—Influence of the hydrogen sulphide content on metal sulphide sols.

Part V.—Influence of temperature: arsenious sulphide and gold sols.

Part VI.—An experimental test of Smoluchowski's theory.

Part VII.—An investigation on the measurement of the electric charge of colloidal particles by the U-tube method. The measurement of the charge by an improved method. The influence of "age" and of dilution on the electric charge.

Part VIII.—Discussion. A connected account of the process of precipitation.

II. Theoretical. The Adsorption of Ions.

The cause of the precipitation by an electrolyte is generally admitted to be due to a decrease in the electric charge resulting from

the adsorption of oppositely charged ions. The nature of this adsorption is not definitely known. An attempt has been made to consider the rôle of the electric forces under the heading :

"The origin of the charge of a colloidal particle and its neutralisation by electrolytes." This paper was read at the discussion on colloids held by the Faraday and the Physical Societies of London, on 25th October, 1922.

PART I.

Experimental Procedure for Determining the Coagulating Power of an Electrolyte or the Rate of Coagulation.

The Ultramicroscopic Method.—Obviously the most direct method is to record by ultramicroscopic measurements the actual progress of the coalescence of the particles. Unfortunately, due to the limitations of the ultramicroscope such measurements are possible in an extremely limited number of cases. It is only recently that Zsigmondy (*loc. cit.*) has been able to improve the method sufficiently to follow the rate of decrease of the primary green particles in his gold sols. The experimental difficulties that come in are that most suspensoids (with the probable exception of two specially prepared sols; gold sols prepared by the nucleus method of Zsigmondy, and the colloidal sulphur of Sven Oden) contain particles of various sizes. Some of these particles are amicroscopic, *i.e.*, they are beyond the range of visibility of the instrument. Galecki (*loc. cit.*) found that as coagulation proceeds the total number of observed particles at first increases due to the formation of visible particles by the aggregation of amicros. The total number of the particles thus remains undetermined.

In order that the measurements can be relied upon the light scattered by each particle must separately reach the eye. Also the intensity of this light must be sufficiently strong. The intensity of the scattered light is dependent on the optical properties of the substance and this introduces a limitation. To prevent the masking of one particle by another, the sol is sufficiently diluted. One cannot always be sure that the dilution itself does not change the original total number of particles. The experiments of Coward (Trans. Faraday Soc., 1913, 9, 1212) may be cited as example. When an electrolyte is present, the addition of a protective colloid is necessary to avoid any further change in the stage of coalescence (*cf.*

Zsigmondy), and also to stop any disintegrating effect that dilution itself may have. But there is no means to ensure that the dilution is not producing any change in the total number. Besides the protective colloid scatters some light and when the light scattered by the particles of the sol is not sufficiently strong it becomes difficult to count the particles.

In the few instances where this method has so far been successfully used, the concentration and the degree of dispersion of the colloid can be varied only within narrow limits. For this reason these colloids are not suitable for investigation on the effect of the quality of the sol on the rate of coagulation.

Other Methods.—There are several other methods of measuring the coagulating power of an electrolyte. The general principle is to find the limiting (minimum) concentration of an electrolyte that will lead to a complete separation of the colloid. Bodlander (Zsigmondy, *Kolloidchemie*, 1920, 66) called this the "Schwellenwert." Linder and Picton (*J. Chem. Soc.*, 1892, 61, 114, 172; 1895, 67, 53) titrated a definite volume of a sol with an electrolyte solution, and observed the amount of the latter necessary to render the supernatant liquid completely clear. Different amounts of electrolyte is, however, necessary, when different concentrations of electrolyte are employed. This is a great objection to this method.

Freundlich (*loc. cit.*) generally mixes a definite volume of the colloid, say 10 c.c., with a definite volume of the electrolyte solution (2 c.c.) of various concentrations. He takes the concentration required to produce the complete precipitation of the colloid in a given interval, as a measure of the coagulating power of the electrolyte. This modified form of the method of determining a "limiting concentration" has been used by several other workers. Two objections can be raised against this method. Firstly, the variation in the limiting concentration produced by a change in the conditions of experiment (*e.g.*, the quality of the sol) is small. This is due to the enormous influence that small changes in the concentration of an electrolyte has on the rate of coagulation. Secondly, much depends on how the limiting concentration is defined. In many instances, the time required for the complete separation of the colloid is very large. When the colloid content of the sol is small it may take several hours for this separation. Thus Young and Neal (*J. Physical Chem.*, 1917, 21, 14) allowed twenty-four hours for this separation. In an investigation on cuprie

sulphide hydrosol by this method they remark that "the amount of electrolyte required is independent of the dilution of the sol within wide limits. This latter was found to be true by Freundlich within rather close limits for arsenious sulphide sols." It will be seen from the sequel that this is erroneous and that this method gives wrong results.

Paine (Proc. Camb. Phil. Soc., 1912, 16, 430) tried to measure the rate of coagulation by estimating the amount of colloid remaining in suspension after different intervals of time. He considers the portion of the colloid that separates easily as the coagulated amount. To hasten this separation he would heat the sol. He found that this heating (up to 100°) does not change the rate of coagulation. It will be seen from the sequel that this is not true.

Lottermoser (Koll. Zeitsch., 1914, 15, 145) has tried to utilise a variation in the physical properties to study the process of coagulation. Freundlich and Ishizaka (Z. Phys. Chem., 1913, 83, 97; 85, 398) utilised the variation in viscosity to measure the rate of coagulation of ferric hydroxide sols. Gann (Koll. Chem. Beihefte, 1916, 8, 113) has carried out an extensive investigation on the precipitation of aluminium hydroxide sols by measuring the change in viscosity.

Method adopted for work.—Most of these colloidal solutions change from day to day. In order to avoid the disturbing effect of this change the method should be comparative and rapid.

It appeared necessary first of all to examine in detail what happens on the addition of an electrolyte to a sol. The precipitation is evidently due to the uniting together of the particles of the colloid, which were so long existing separately to form bigger aggregates (clots and flakes). The subsequent settling of these aggregates is a distinctly different process and it is important to know what part these two processes play in a particular experimental method. An examination of the process of coagulation made it clear (Mukherjee, J. Amer. Chem. Soc., 37, 2024-2026 (1915); Reprint No. 2 attached herewith) that "the settling of the particles or clots is irregular. The times for its complete settling is not characteristic of the rate of coagulation; but that required for the appearance of perceptible change or for the attainment of the maximum opacity or for the beginning of the settling yields a definite idea of the rate of the change." In a later paper (Mukherjee and Sen, Trans. Chem. Soc., 115, 1919, 462—Reprint No. 3 attached herewith) the objections against the "limiting

concentration" methods as adopted by Freundlich and others have been stated as follows: "The justification of any method used to follow the coagulation lies in so far as it indicates the progress of coalescence. The increased mass of the particles with progress of coalescence introduces a new factor; namely, their gravitational effect, which masks the true behaviour of the sol, as will be clear from the following observations on mercuric sulphide sols.

These sols are opaque unless very dilute. On the addition of electrolytes, there is a quiescent period followed by a sharp clearing of the whole liquid. At this stage, the liquid loses its homogeneous appearance, and visible clots are found suspended throughout the liquid. As the change is sharp, the times noted by different observers agree satisfactorily. In this way, it is found that a mercuric sulphide sol, on saturation with hydrogen sulphide, requires a longer time for the observed change than when it is not so treated—the electrolyte concentration, of course, being identical.

These experiments leave no doubt that dissolved hydrogen sulphide increases the stability of the sol. The subsequent settling of these clots, however, requires a very long time, which is about the same for both samples. So long as the respective times required for the clearing of the sols are very small compared with the time required for the subsequent settling of the clots, it is found that the times that are necessary for the complete separation of the colloid do not differ much in the two cases. However, with electrolyte concentrations where the clearing requires intervals comparable with that required for the subsequent settling of the clots, regular differences in stability are observed even if the times necessary for complete separation of the colloid are noted." Similar observations have been made on gold sols.

The method ultimately adopted is a comparative one and consists in observing what change the mixture of sol and electrolyte undergoes with time when equal volumes (generally 5 c. c.) of both are mixed together. The comparison is made with reference to a blank containing equal volumes of water instead of the electrolyte, to determine the first perceptible change in the mixture. To secure a constant manner of mixing the electrolyte is always added to the sol. Five c. c. of each are measured out into carefully cleansed and dried test-tubes of as nearly the same size as it is possible to get. The time when the sol attains a limit of opacity is next noted. This is usually done by observing the time when the sol looks completely opaque

when held in front of the eye against the window, or an illuminated plate of glass or a filament of an electric light according to what is most suitable. If the sol is not too rich in its colloid content the breaking up of the sol into visible clots suspended in a clear liquid gives a sharp indication and is taken as the next stage of comparison. A little shaking of the tube by the hand helps the breaking up of the colloid into clots. This does not interfere with the accuracy to any great degree.

The process of coagulation of these suspensoids is irreversible and the addition of small quantities of an electrolyte of much higher concentration cannot fail to produce immediate changes, which the end concentration of electrolyte produced by the mixture is of itself not able to effect. This is more objectionable if the same quantity of the salt is added in a small volume, as it necessarily means a higher initial concentration of the electrolyte; for, with rise in concentration of the electrolyte the rate of coagulation increases enormously. On the other hand, on mixing the sol with the electrolyte the sol always becomes diluted depending on the ratio of the volumes of the two liquids mixed. To avoid too great a dilution equal volumes of electrolyte and of sol are mixed.

The "times" which are noted change very rapidly with slight changes in the experimental conditions and the sensitivity of this procedure is due to this fact. The method is simple and rapid. In general three to four readings are taken. The agreement is quite satisfactory. It was not considered necessary to use better methods of comparison, for example the use of a nephelometer, as this rapid method is sufficiently accurate for the purpose. Electrolytes are generally used in such concentrations as would give fairly rapid rates of coagulation so that the observed changes are sharp. It has been found that readings within three or four days are quite comparable. Evidence as to whether the sol has changed or not has been directly obtained by determining the mean value of the "time" for a particular change under definite conditions. A variation in the time outside the limits of error shows that the sol has changed.

Preparation of Arsenious Sulphide Sols of Different Quality.

The mode of preparation of this sol is well known and only a few remarks will be made here. The finest sols are obtained when a

large volume of pure water is thoroughly saturated with pure hydrogen sulphide and a dilute solution of molecularly dissolved arsenious oxide (that is, free from suspended particles of arsenious oxide) is added to it. The two solutions should be mixed as rapidly as possible and care should be taken that the hydrogen sulphide is much in excess of that required to form the sulphide. Perfectly clear solutions can be obtained in this way. They look reddish-brown in transmitted light whereas coarser sols look more yellowish. The sol is next saturated with hydrogen sulphide by passing a rapid stream of the pure gas for half an hour. The dissolved hydrogen sulphide is removed by a current of pure hydrogen.

The arsenious oxide solution is best prepared by saturating pure "conductivity" water with the powdered oxide. The rate of solution is slow and the powder floats as it is not wetted. It is convenient to leave the vessel in contact with a small flame for several hours so that the water is kept boiling very slowly. The liquid should then be allowed to cool and left for 24 hours to allow the excess of arsenious oxide dissolved at the higher temperature to separate completely. Otherwise they pass through the filter and produce big particles. The solution is then filtered.

For preparing sols with different degrees of dispersion the procedure is theoretically the same as that underlying the nucleus method of Zsigmondy. The usual method of preparing the sol consists of two stages :

- (1) the mixing of the hydrogen sulphide solution with the arsenious oxide solution, and

- (2) the subsequent saturation of the mixed liquid with hydrogen sulphide. When conditions are satisfactory the arsenious oxide becomes converted completely into arsenious sulphide in the first process and there remains an excess of hydrogen sulphide after the reaction. In this case fine sols are formed. The subsequent passage of hydrogen sulphide is to prevent any aggregation of the particles. The size of the particles in this case is governed by what happens in the first stage. As there is plenty of dissolved hydrogen sulphide and a very dilute solution of arsenious oxide is mixed with it the probability of the union of the resulting molecules of arsenious sulphide, to form bigger aggregates, is smaller than when the reverse is the case. The size of the aggregates may be said to depend on the concentration of the colloid formed. This subject has been

treated fully in Zsigmondy's book "Kolloidchemie," 1920, pp. 145-148, 152.

When an excess of the arsenious oxide remains unconverted into the sulphide, the particles formed in the first stage, act as nuclei and grow in size. Thus by adjusting the relative volumes of the two solutions and the concentration of the oxide, sols of different quality, can be prepared. Any desirable colloid content can be obtained by adding water after the second process. It is easy to prepare sols with different degrees of dispersion in this way.

PART II.

Influence of the Dilution of the Sol on the Rate of Precipitation.

It is necessary to point out at the outset, that when sols differing in quality are compared, the comparison cannot be absolute. A diluted sol has different physical properties than the original sol. Thus its turbidity as measured by a nephelometer or its absorption coefficients for a definite wave-length of light, is different from that of the original sol. Since in general the rate of coagulation is defined by the time taken to produce a definite change in the sol, the standard of comparison of one sol is arbitrary for the other. The manner in which any of these properties change with the progress of coalescence is very complicated and no relationship between these physical properties and the average size of the aggregates is known. These indirect methods, therefore, lose a quantitative basis of comparison. The quantitative relationship varies according to the conditions utilised for comparison. This fact is sometimes lost sight of in discussing the results of different authors.

At the same time there cannot be any doubt as to the qualitative nature of the effect observed. Thus it was observed on one occasion that with N/16 NaCl, arsenious sulphide begins to separate with a sol in ten minutes, but when the sol was diluted twenty times no separation of the colloid was observed unless full ten days have elapsed.

The following data are taken from Reprint No. 3. (Mukherjee and Sen, T., 1919, 115, 464-6.)

TABLE I.

Sol A contained 17.58 millimoles of arsenious sulphide per litre. Sol B was prepared by diluting Sol A five times, and Sol C by diluting Sol A ten times, with pure water.

Electrolyte: lithium chloride.

Dilution (after mixing).	Sol A.	Sol B.	Sol C.
5N/16	Coagulation after 20 seconds.
5N/32	...	Coagulation after 30 seconds.	Coagulation after 50 minutes.
N/8	Instantaneous coagulation.	Coagulation after half-an-hour.	Perceptible change after 8 minutes. Coagulation after 2 hours 15 minutes.
N/16	Change perceptible on mixing; coagulation not observed after 1½ hours.	Change just after mixing not perceptible.	Change perceptible after 45 minutes.

TABLE II.

Electrolyte: thorium nitrate.

Dilution.	Sol A.	Sol B.	Sol C.
N/10,000	Instantaneous coagulation.	Instantaneous coagulation.	Instantaneous coagulation.
N/20,000	Perceptible turbidity just after mixing. Sol changes slowly.	Coagulation in 2 minutes.	Coagulation within half a minute.
N/30,000	Perceptible turbidity after half-an-hour.	Perceptible turbidity after 5 minutes.	Coagulation in 4 minutes.
N/40,000			Coagulation in 53 minutes.

TABLE III.

Electrolyte: barium chloride.

Arsenious sulphide sol containing 19.45 millimoles per litre.

Dilution of Electrolyte.	Original Sol.	Sol diluted 4 times.	Sol diluted 16 times.
N/800	Complete coagulation in 1 minute.	Change perceptible in 20 seconds. Coagulation after 4 minutes.	Perceptible change after 1 minute. Coagulation after 12 minutes.
N/1,000	Change perceptible after half a minute. Clots appear after 17 minutes.	Change perceptible in 1 minute. Clots appear throughout after 18 minutes.	Change perceptible after 2 minutes. Clots appear throughout after 26 minutes.
N/1,200	Change perceptible in 2 minutes. Clots appear after 1 hour 7 minutes.	Change perceptible in 2 minutes. Clots appear after 52 minutes.	Change perceptible after about 3 mins. Clots appear after 1 hour 1 minute.

TABLE IV.

Electrolyte: aluminium sulphate.

Sol contained 19.45 millimoles arsenious sulphide per litre.

Dilution.	Original Sol.	Sol diluted 4 times.	Diluted 16 times.	Diluted 20 times.
N/4,000	Coagulation in 7 minutes.	Coagulation immediately on mixing.	Coagulation in 40 seconds.	Coagulation in 50 seconds.

A summary of the main results is given below.

(a) The nature of the cation of the electrolyte determines whether the diluted sol is more or less stable than the original sol. The degree of dilution and the quality of the sol has also to be considered.

(b) A diluted sol of arsenious sulphide is mostly more stable than the undiluted sol when the precipitating cation is univalent.

(c) A diluted sol of arsenious sulphide is always less stable when the cation is aluminium (trivalent), or thorium (tetravalent).

(d) When the cation is divalent, either a diminution or an increase in stability may be observed. At moderate dilution the sol becomes unstable. At higher dilutions the sol may become stable depending on the degree of dilution. Somewhat similar results may be observed with aluminium sulphate.

(e) Cupric and mercuric sulphide sols always showed an increase in stability on dilution. This is connected with the relatively high concentration of the electrolytes necessary to precipitate them, and the smallness of their colloid content and the lower degree of dispersion.

The Effect of the Amount of Colloid-liquid Interface on the "limiting concentration."

These observations afford a clue to an understanding of the opposite effects of dilution recorded by different observers. Freundlich in his experiments restricted himself to trivalent cations and hence observed that with dilution of the sol the limiting concentration of the electrolyte required to coagulate an arsenious sulphide sol decreases. He explained this on his adsorption theory as follows. On dilution ¹ the number of particles of the colloid per unit volume decreases. Consequently the extent of the colloid-liquid interface also decreases. The amount adsorbed per unit area depends on the equilibrium concentration. Let x_u and x_d be the amount of cations adsorbed per unit area of the interface for the undiluted and diluted sols respectively. If s_u and s_d be the corresponding colloid-liquid interface per unit volume of the two sols and " c " the amount of the cation present in unit volume then the respective equilibrium concentrations γ_u and γ_d (that is the amount of cation remaining in the solution per unit volume after adsorption) are given by

$$\gamma_u = c - x_u \cdot s_u \dots \dots \dots (1)$$

$$\gamma_d = c - x_d \cdot s_d \dots \dots \dots (2)$$

Since $s_u > s_d$ we have $\gamma_u < \gamma_d$; i.e., the equilibrium concentration is greater when the sol is dilute.

¹ Throughout this discussion it is assumed that the size and properties of the particles do not change on dilution.

According to Hardy coagulation takes place at the iso-electric point. Freundlich concludes that since the iso-electric point depends on the equilibrium concentration and has been reached when it has the value equal to γ_u the limiting concentration for the dilute sol will have a total concentration equal to

$$c' = \gamma_u + x_u \cdot s_d \dots \dots \dots (3) \bullet$$

It is obvious that c' is less than c . It should be noted that on this view the difference between c and c' is given by

$$c - c' = x_u \cdot (s_u - s_d) \dots \dots \dots (4)$$

This view of the process therefore predicts that on dilution a sol should always be less stable. Considering the smallness of the amount adsorbed it is easy to understand that a variation in the amount adsorbed will make its effect felt only when the total amount of electrolyte present is comparable to the amounts that are adsorbed; that is, $s_u \cdot x_u$ is comparable in magnitude to " c ." The electrolytes with trivalent and tetravalent cations can coagulate negatively charged suspensoids at very low concentrations and it is in these instances that the above effect is most pronounced. On the other hand when the concentration of the electrolyte is so large, e.g., N/10 NaCl, that $x_u \cdot s_u$ or $x_u \cdot s_d$ are negligible compared to " c ," then this effect will be absent. *That is, the diminution in stability will not be observed and both sols should be equally stable.* The increase in stability on dilution is not contemplated in this theory.

The Influence of the Distance between the Particles in Determining the Number of Collisions.

The opposite effects of polyvalent and univalent cations strongly resemble similar effects noticeable in biological systems. These apparently contradictory factors can, however, be easily understood if one considers the influence of the distance between the particles of the colloid. In order that two particles may coalesce they must meet. The rate of collision is determined by their rate of diffusion resulting from their Brownian movement, and the distance they have to cover before they can meet. On diluting the sol this distance increases. The dependence of the rate of collision on the number and the rate of diffusion has been clearly formulated by Smoluchowski.

Thus on diluting a sol two opposite factors are brought into play. Firstly, a decrease in the surface which diminishes the stability and secondly, an increase in the distance between the particles or a decrease in the rate of collisions, which increases the stability. The observed results are, therefore, due to the combined effect of these two factors. For arsenious sulphide sol the concentration of the cations of different valencies that produce instantaneous coagulation has roughly the following values.

TABLE V.

Valency of cation.....	1	2	3	4
Concentrations.....	N/10	N/600	N/6,000	N/10,000

It is apparent that the influence of a variation in the amount absorbed will be negligible with univalent cations and the effect of dilution preponderates. A balance between the two effects have been observed with barium chloride and aluminium sulphate (Tables III and IV). Kruyt and Spek (Proc. K. Akad. Wete. Amsterdam, 1915, 17, 1158) who also found an increase in stability on dilution for potassium could not observe the same with barium. This is probably due to their not varying the dilution sufficiently. Recently Burton and Bishop (J. Phys. Chem. 1920, 24, 701) have studied the coagulation of sols of arsenious sulphide, mastic and copper, and have come to the same results as recorded above (*cf.* also Kruyt and Arkel. Tec. Trav. Thim. Pays-Bas, 39, [IV], 1920, 509).

Evidence of a Continued Variation in the Percentage of Successful Collisions beyond the "Limiting Concentration."

One important conclusion can be drawn from the stabilising effect of dilution observed with univalent cations. This is that the older view that coagulation takes place at the iso-electric point is untenable. Let us assume that the original undiluted sol coagulated immediately on mixing with the electrolyte when the end concentration is N/16 of sodium chloride. A sol prepared by diluting it four times is not coagulated immediately when the electrolyte has the same concentration. It is, however, coagulated immediately when the concentration is N/14. It is evident that the iso-electric point must have been

reached when the end concentration is $N/16$ for both sols. Of course it is assumed that the particles do not change on dilution. It will be seen in the sequel that on dilution the velocity of migration in an electric field remains constant within the limits of experimental error. There is thus no reason why the diluted sol should not coagulate at the same electrolyte concentration.

The fact that at a higher concentration of the electrolyte the coagulation is instantaneous shows that the rate of coalescence or the rate of successful collisions can be extremely great. The total number of encounters between particles per unit volume in unit time is constant and does not depend on the electrolyte concentration; for it can be safely assumed that the Brownian movement remains unaffected by the electrolyte concentration (*cf.* Svedberg, *Die Existenz der Moleküle*, 1921, p. 105). It is therefore necessary to assume that the higher concentration of the electrolyte only increases still further the percentage of the total number of collisions that produces a stable coalescence.

Similarly the view that coagulation takes place only at a definite critical potential of the double layer is untenable. One is forced to admit that the percentage of successful collisions has not reached its full value (hundred) at the limiting concentration and that it increases with higher concentration. This increase cannot be observed because the rate of coalescence has already become practically infinite. On dilution the total number of collisions decreases and to increase the rate of coalescence to its former value a greater percentage of successful collisions is necessary. The higher electrolyte concentration is necessary for this effect.

PART III.

Influence of the Quality of the Sol.

On dilution of a sol the distance between the particles increase at a rate proportional to the cube root of the dilution, but the colloid-liquid interface per unit volume decreases at a rate proportional to the dilution. An increase in the distance and a decrease in the surface have opposite effects. It would, therefore, be interesting to observe the effect of varying these two factors simultaneously in the same ratio. Two sols which have the same mass of colloid per unit volume but have different numbers of particles satisfy this condition. It is to be

expected that in this case the influence of a variation in the distance will be more prominent than in the other case. Care was taken to maintain the hydrogen sulphide content constant. Both sols were prepared on the same day.

It was found that the finer sol is less stable with univalent and divalent cations. The variation in surface was insufficient to counteract the influence of that in the distance even for divalent cations.

A simple calculation will show that both the distance and the surface varies in the same ratio. Since the total amount of the colloid per unit volume is the same we have

$$n_1 v_1 = n_2 v_2 \dots\dots\dots (5)$$

where v_1 and v_2 are respectively the average volumes of the particles in the finer and in the coarser sol; n_1 and n_2 are the corresponding numbers of particles in unit volume.

The mean distances between the particles vary as

$$r_1 : r_2 = (n_2)^{1/3} : (n_1)^{1/3} \quad (6)$$

The colloid-liquid interface per unit volume may be taken as proportional to

$$n_1 \cdot v_1^{2/3} \text{ and } n_2 \cdot v_2^{2/3} \text{ respectively.}$$

From (5) we have

$$n_1/n_2 = v_2/v_1$$

$$\text{or } (n_1/n_2)^{2/3} = (v_2/v_1)^{2/3} \quad (7)$$

The ratio of the surfaces S_1 and S_2 respectively is given by

$$\begin{aligned} S/S_2 &= \frac{n_1 v_1^{2/3}}{n_2 v_2^{2/3}} = \frac{n_1}{n_2} \left(\frac{v_1}{v_2} \right)^{2/3} \\ &= \left(\frac{n_1}{n_2} \right)^{1/3} \end{aligned} \quad (8)$$

Thus both factors change in the ratio $(n_2)^{1/3} : (n_1)^{1/3}$ but in the opposite direction.¹ Since $n_1 > n_2$ the finer sol has a larger value for the surface and smaller distance than the coarser sol. The greater surface tends to make it stable and the smaller distance

¹ The average time required for one particle to meet another (by diffusion) may be taken to vary directly as the distance between them. The rate of collisions would thus in the initial stage vary in the ratio $(n_1)^{1/3} : (n_2)^{1/3}$

tends to make it unstable. The effects are thus opposite. The following data are taken from Reprint No. 3.

TABLE VI.

Electrolyte : strontium chloride.

Both sols contained 8.52 millimoles of arsenious sulphide per litre.

Dilution.	Fine Sol. (I)	Coarse Sol. (II)
N/200	Coagulation after a few seconds	Coagulation after a minute
N/300	Coagulation after two minutes	Coagulation after four minutes.
N/400	Coagulation after thirteen minutes.	Turbidity perceptible after 1 minute. A portion of the colloid had separated after 40 minutes.
N/500	The greater portion had separated after 40 minutes.	

It appears that the magnitude of the difference in stability is roughly the same for the different electrolytes. It will be seen from the sequel that dissolved hydrogen sulphide has an anomalous effect on the rate of coagulation of arsenious sulphide sol in the case of certain salts. Here, also, the greater stability of the coarser sol is marked. In the next table are given the respective concentrations of an electrolyte which corresponds with about the same coagulation time for these two sols.

TABLE VII.

Electrolyte.	Comparable Concentrations.		Remarks.
	Sol I.	Sol II.	
Strontium Chloride	N/500	N/400	In presence of H ₂ S.
Do	N/400	N/300	
Lithium Chloride	N/8	5N/32	
Ammonium Chloride	N/20	N/16	
Potassium "	N/20	N/16	

The influence of the degree of dispersion is quite pronounced. It may be pointed out that the mere mention of the colloid content per unit volume means very little in defining its properties. From the following section it will be seen that the hydrogen sulphide content of a sol has also a marked influence.

PART IV.

Influence of the Hydrogen Sulphide Content on Metal Sulphide Sols.

It is well known that hydrogen sulphide is necessary for the peptisation of the precipitated sulphide. It is, therefore, natural to expect that when a sol is saturated with hydrogen sulphide it will be more stable than when it is free from hydrogen sulphide. This is not so. It was at first observed that the stability increases when the precipitation is carried out with electrolytes having univalent cations. Later, it was found that the opposite effect is observed with bivalent cations. Table VIII shows the concentrations of the electrolyte which produce roughly the same rate of coagulation in the sol in the presence and in the absence of hydrogen sulphide. The results show that the magnitude of the effect varies somewhat with the quality of the sol used.

TABLE VIII.

Comparable Concentrations.

Electrolyte.	Sol. I. H ₂ S		Sol. II. H ₂ S		Sol. III. H ₂ S	
	Absent.	Present.	Absent.	Present.	Absent.	Present.
KCl ...	N/18	N/8	N/16	N/8
NH ₄ Cl ...	N/20	N/12	N/20	N/12	N/20	N/12
BaCl ₂ ...	N/800	N/1000	N/800	N/1000
SrCl ₂ ...	N/300	N/400	N/300	N/400
Th(NO ₃) ₄ ...	N/10000	N/12000

A summary of the results is given below.

1. Arsenious sulphide sol.

- (a) In the presence of hydrogen sulphide the sol is more stable, when the coagulating cation is univalent (K⁺, NH₄⁺, Na⁺).
- (b) The reverse is the case when the cation is bivalent (Ba⁺⁺, Sr⁺⁺).

- (c) With trivalent or tetravalent cations the influence is irregular. The nature of the change depends on the concentration of the electrolyte and on the quality of the sol. These electrolytes are hydrolysed in dilute solutions and their behaviour is complicated (*cf.* Powis). More elaborate work with these electrolytes is desirable; though, because of the extremely complex nature of the phenomena, a satisfactory solution can only be expected after the simpler cases have been clearly understood.

2. *Mercuric Sulphide.*

- (a) Increased stability in presence of hydrogen sulphide observed with the chlorides of ammonium and potassium.
- (b) Diminution in stability with the chlorides of barium and strontium. That is the same effect as with arsenious sulphide.

3. *Copper Sulphide.*

A diminution in stability was observed with the chlorides of potassium, ammonium, strontium, barium and the sulphate of aluminium.

It seems that there are several factors to be taken into account. A removal of dissolved hydrogen sulphide probably produces several simultaneous changes in the sol, namely, the electric charge of the particles and the degree of dispersion. Young and Neal (*loc. cit.*) observed that with increasing amounts of hydrogen sulphide, the velocity of migration of the particles in an electric field increases, reaches a maximum and then falls. The degree of dispersion seems to undergo the same changes. The increase in the charge should make the sol more stable. On the other hand, Young and Neal observed that in the presence of hydrogen sulphide the rate of migration of particles of a copper sulphide sol decreases. This decrease would lead one to expect the diminution in stability of the sol that has been observed. The method of measuring the charge of the particles from the rate of motion of the colloid boundary, as used hitherto, is not accurate and there are some serious objections against the soundness of the usual method. This subject will be dealt with in a separate section.

In the absence of any reliable data on the changes produced in the sol by the removal or introduction of hydrogen sulphide it is not possible to definitely account for the differences observed. These observations show the complexity of the process of precipitation and clearly contradicts the very common belief that a peptising substance increases the stability against precipitation. It is, so far as the writer is aware, *the only case where the same substance acts as a protective agent on a suspensoid with regard to one electrolyte and acts in the opposite way in the case of another.*

One definite conclusion can be drawn from the observations recorded in Sections II, III and IV. *It is that a quantitative comparison of the coagulating powers of the different electrolytes as measured by their coagulating concentrations is impossible and faulty.* Thus the ratio of the concentrations of four cations having different valencies which produces the same rate of coagulation varies with the quality of the sol and the amount of hydrogen sulphide present. Many of the discussions on this subject would have been avoided if the above facts were more generally known.

Influence of alkali sulphides.—It is interesting to note in this connection that traces of alkali or alkali sulphides have a more marked protective action on sulphide sols. (Reprint No. 3, p. 471.) These substances dissolve arsenious sulphide forming arsenites and thioarsenite. So the observed protective action cannot be referred to any particular ion. Indeed, the liquid obtained by dissolving in a few cubic centimetres of dilute alkali as much arsenious sulphide as possible has an equally marked protective action on the sols of these three sulphides. The stabilising effect has been observed with the chlorides of ammonium, potassium, barium and strontium, and with the sulphate of aluminium.

It was next thought desirable to examine the influence of temperature on the process of precipitation.

PART V.

Influence of Temperature.

There are very few observations on the effect of variations in the temperature on the rate of coagulation.

The need of such investigations for an understanding of the mechanism of the process is obvious. The precipitation of arsenious sulphide sols of various qualities have been examined with the following electrolytes at temperatures lying between 0°C and 65°C:—

Chlorides of potassium, sodium, lithium; nitrate and sulphate of potassium; sulphuric and hydrochloric acids; the chloride of barium and strontium; the chloride and the bromide of calcium; aluminium sulphate and thorium nitrate.

It was first ascertained that the sol does not suffer any irreversible change with a rise in the temperature. The experimental data are given in Reprint No. 4. The influence of temperature is quite complex. Whether an increase or a decrease in the stability is observed depends on the nature of the electrolyte, the range of variation of the temperature and the quality of the sol. The electrolytes are classified below according to their behaviour.

1. Electrolytes that invariably show an increase in stability at higher temperatures—the salts of alkali metals fall in this group.

2. Electrolytes which invariably show a decrease in stability with rise in temperature—the two acids and the salts of the divalent cations.

3. Electrolytes which show a more complex behaviour depending on the range of temperature used and on the quality of the sol—aluminium sulphate and thorium nitrate.

In the following tables some of the observations are given as illustrations. The times recorded are strictly comparable only for a particular sample at different temperatures. The data are taken from Reprint No. 4 (Mukherjee, T., 1920, 117, pp. 350-358).

The observations at different temperatures were made side by side and always on the same day. The sulphide contents of the different sols are given below. Sol A contains 1.925 millimoles of arsenious sulphide. Sols B, C and F have the same sulphide content, namely, 14.23 millimoles per litre. Sols D, E and I contain 3.5575 millimoles of sulphide per litre, but Sol E contains an equal amount of arsenious oxide in excess. Sols H and J each contain 4.8 millimoles per litre. In comparing the different sols, it should be remembered that the behaviour of a sol is not determined by its sulphide content alone.

TABLE IX.

Electrolyte.	Concentration.	Sol used.	Times required at different temperatures.	
			25°.	45°.
1. Potassium chloride ...	17N/200	A.	4 mins.	38 mins.
2. " "	N/20	B.	14 "	38 "
3. " "	N/18	B.	4 "	10 "
4. " nitrate.	N/20	C.	1 hr. 51 mins. (14.5°)	3 hrs. 39 mins.
			14.5°.	45°.
5. Potassium sulphate ...	N/7	D.	3½ mins.	4 hrs. 2 mins.
6. " "	N/7	E.	1 min.	20 mins.
7. Sodium chloride ...	3N/40	F.	10 mins.	4 hrs. 42 mins.
8. Hydrochloric acid ...	0.07385N	D.	5 mins.	About 1 min.
9. " "	0.07385N	E.	4 "	" 1½ "
10. Sulphuric acid ...	0.08438N	E.	40 "	23 mins.
11. Hydrochloric acid ...	0.0733N	F.	11 "	3 "
			28°.	65°.
12. Barium chloride ...	N/800	H.	5 mins.	1 min.
13. Calcium bromide ...	N/800	H.	13 "	2 mins.
14. Strontium chloride ...	N/800	H.	21 "	4½ mins.
15. Magnesium sulphate ...	N/600	H.	7 "	2 "

Observations 1 and 2, 1 and 3, and 5 and 6 show clearly that the effect of temperature depends on the quality of the sol. The concentration of the electrolyte has also an effect even for the same sol.

It would appear from Table IX that the extent to which the time varies with change in temperature differs with different preparations of the sol. For any particular sol, however, the effect of temperature is approximately the same for the different ions within the groups referred to above.

Experiments were made with freshly prepared sols with excess of arsenious oxide, and the results were similar. In all, fifteen different samples have been used, and the results were always concordant. The experiments with excess of arsenious acid show that the effect of temperature cannot be wholly referred to hydrogen sulphide evolved by the decomposition of arsenious sulphide. It is true that rise of temperature increases the rate of decomposition of arsenious sulphide, but the quantity produced in this way is extremely small. Moreover, the rate of evaporation of dissolved hydrogen sulphide also increases. It would appear from what follows that the effect of temperature is rather complex, and a slight change in the quantity of hydrogen sulphide cannot explain it. It may be pointed out that freshly prepared sols have no perceptible odour of hydrogen sulphide at any of the temperatures investigated, and that the magnitude of the effect observed would require a rather marked change in the concentration of dissolved hydrogen sulphide.

With aluminium sulphate and thorium nitrate, more complicated results were obtained. In some cases, an increase of temperature increased the time, and in other cases the reverse is the case. With a particular sample of the sol and a particular concentration of electrolyte, different readings agreed with one another perfectly. On closer investigation, it was found that concentration of the electrolyte, as well as the quality of the sol, have to be taken into consideration. It must also be remembered that a dilute solution of aluminium sulphate "ages" considerably with time, as pointed out by Powis (*Zeitsch. Physikal. Chem.*, 1914, 89, 106). The combined effect of "ageing" of the sol and the electrolyte will be apparent from Table X, in which the time for coagulation diminishes with 1.064N/9000 aluminium sulphate from six and a half minutes to two and a half minutes eleven days later, but the influence of temperature is the same in both cases.

TABLE X.
Electrolyte: Aluminium Sulphate.

Concentration.	Sol.	Corresponding times at different temperatures.		
		Room temperature.	25°	45°
1.064 N/7000 ...	I	3 mins.	2 mins.	1½ min.
1.064 N/9000 ...	I	6½ "	...	More than 12 min
1.064 N/9000 (11 days later)		2½ "	...	4 mins.
1.064 N/10000 ...	I	6 "	...	17 "
1.064 N/10000 ...	D	15 "	...	3 hrs. 28 mins.
N/5000 ...	J	2 hrs. 41 "	57 mins.	1 hr. 26 mins.

Table XI records some similar data with thorium nitrate. Similar irregularities will be noticed here. It would seem that a change in temperature produces independent changes in several factors which govern the mechanism of coagulation.

TABLE XI.
Electrolyte: Thorium Nitrate.

Concentration in grams of the crystallised salt per litre.	Sol.	Corresponding times at different temperatures.		
		4°.	28°.	60°.
0.025 ...	A	4 mins.	1 min. 80 secs.	more than 2 hrs.
0.02 ...	A ₁	11 "	28 mins.	Not in 12 hrs.
0.0175 ...	A ₁	81 "	Not in 2 hrs.	...

Sols A and A₁ are of the same sulphide content, but prepared on different dates.

Experiments with Sols containing Ethyl Alcohol and Phenol.

Kruyt and Duin (Koll. Chem. Beihefte, 1914, 5, 269) investigated the effects of adding substances like Ethyl alcohol, isoamyl alcohol, and phenol, which diminish the surface tension (air-liquid surface) of water. They used them in concentrations varying from N/50 to

3N/10. It is needless to point out that in such concentrations most of the physical properties of the dispersion medium will be altered. Their work has an interesting bearing on the results recorded here in so far as they also found that the bivalent group of cations act in a different manner from the uni- and ter-valent groups. In fact, they observed the reverse effect, namely, an increase in stabilisation with the barium group and a decrease with the rest.

Experiments at three different temperatures with potassium chloride, barium chloride, strontium chloride, calcium bromide, magnesium sulphate, aluminium sulphate, and thorium nitrate were made with sols containing small quantities of alcohol and phenol.

To a definite volume of a sulphide sol was added a definite volume of a solution of alcohol or of phenol of known strength, and in a blank experiment water was added in the same ratio.

With potassium and the bivalent cations exactly the same results were obtained as with the ordinary sols, that is, an increase in stability with rise of temperature for potassium and a decrease in the case of bivalent cations. The concentration of phenol in the sol was about 0.515 gram in 100 c.c., and that of alcohol varied from 5 c.c. to 25 c.c. of absolute alcohol in 100 c.c. of the sol. With the small quantities of phenol used, no noticeable change was observed from the blank experiment, but with alcohol a slight decrease in stability for potassium and a slight increase with the bivalent cations were noticed. This is in agreement with the observations of Krøyt and Duin (*loc. cit.*), but the magnitude of the change is small.

With aluminium sulphate and thorium nitrate, more complicated results are obtained, as is to be expected from the temperature-effect with the pure sols, and they will be dealt with on a future occasion.

A few selected data are given in Table XII.

TABLE XII.

Electrolyte: Strontium Chloride. Concentration, N/800.

		Corresponding times.		
		0.2°.	28°.	65°.
Sol with water	...	1 hr. 15 mins.	21 mins.	4½ mins.
Alcohol present	...	"	30 "	4½ "
Phenol "	...	"	20 "	3½ "

Calcium Bromide, N/800 approximately.

Sol with water	...	1 hr. 18 mins.	10 mins.	2 mins.
Alcohol present	...	1 hr. 23 mins.	22 "	2 "
Phenol "	...	1 hr. 15 mins.	13 "	About 2 mins.

Magnesium Sulphate, N/600.

Sol with water	...	35 mins.	7 mins.	2 mins.
Alcohol present	...	34 "	7 "	1½ "
Phenol	...	33 "	6 "	About 2 mins.

The hydrosol contained 4.8 millimoles of sulphide per litre.

From the above it would appear that a variation in the temperature produces independent changes in several factors which govern the mechanism of coagulation. A variation in the temperature may have the following effects:—

- A. A change in the degree of dispersion.
- B. Increased Brownian movement of the particles due to greater thermal energy and smaller viscosity.
- C. An increase in the potential of the double layer surrounding the particles due to a decrease in its dielectric constant—on the assumption that water molecules are present between the two layers in sufficient thickness, the dielectric constant can be taken to be that of water in bulk as is done usually.
- D. A change in the electric charge of the particles which will also vary the potential of the double layer.
- E. A variation in the adsorbability of the precipitating ion; that is, a variation in its precipitating power.

The influence of temperature on the Brownian movement and on the viscosity is definitely known. It would increase the rate of collisions at a rate depending on the value of θ/η where " θ " denotes the absolute temperature and " η " the viscosity; for according to Einstein D, the rate of diffusion of particles executing Brownian movement is given by

$$D = \frac{R\theta}{N_0} \cdot \frac{1}{6\pi r\eta} \quad (9)$$

where R =the gas constant in C.G.S. units; N_0 is Avogadro's number, and r denotes the radius of the particles.

The influence of the variations mentioned in C and D can be ascertained experimentally by measuring the migration velocity in an electrical field at different temperatures. The changes considered under A, B, C and D are common to all electrolytes. The fact that an increase in the stability is observed with potassium salts and that a decrease in stability is observed with hydrochloric acid, enables us to conclude that the difference cannot be due to any difference in the condition of the sol. For, in this case, the change in the degree of dispersion would act in the same way for both. We have also seen that a difference in the degree of dispersion acts in the same way with both univalent and divalent cations. A change in the electrical potential of the double layer would affect the stability equally in each case. We are thus left with one variable, namely, E, to account for the different effects of the various electrolytes that have been studied.

The adsorbability of the various cations thus changes in a different manner in each group with a variation in the temperature.

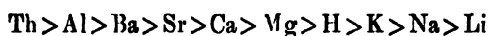
Freundlich and Ishizaka (Z. Phys. Chem., 1913, 83, 97 ; 85, 398) and Freundlich and Gann (Koll. Chem. Beihefte, 1916, 8, 67) have suggested that the process of precipitation is autocatalytic in nature. The influence of temperature recorded above shows that the kinetics of precipitation cannot be compared with the kinetics of chemical reactions where a much higher increase in the rate of reaction is usual.

Experiments with gold sols.

In the case of gold sols the experiments of Zsigmondy have shown that there is no change in the degree of dispersion on boiling the sol. The time taken for the change of the colour of the sol to a certain nuance of bluish-violet can be observed very accurately. The experimental data are given in Part VI. Precipitation by the following electrolytes has been studied between 15°C to 50°C:—the chloride and sulphate of potassium; barium chloride and strontium nitrate. With potassium chloride as the precipitating agent both an increase and a decrease in the stability has been observed depending on the temperature. In all other cases a decrease in stability has been observed. Here also a variation in adsorbability with temperature has to be assumed. The subject will be discussed from the point of view of Smoluchowski's theory in the next section.

A General Conclusion.

The previous sections will show the complex nature of the process of precipitation and the futility of quantitative comparison of the coagulating power of electrolytes from a determination of the concentration which produces the same rate of coagulation. It is, however, interesting to note that throughout the influence of the valency of the cation predominates. The *order* of the coagulating power of the different electrolytes is determined by the cation and is as follows :



independent of changes in the sol.

It may be pointed out that the present work has been restricted to the simpler inorganic electrolytes.

PART VI.

Experimental Test of Smoluchowski's Theory of the Kinetics of the Process of Coagulation.

A Short Account of the Theory. In the experiments on the degree of dispersion of colloidal arsenious sulphide on the rate of coagulation it has been shown that the finer sol is less stable. In 1915 the writer pointed out the obvious connection with the increased facilities of coalescence. The smaller particles have a more vigorous Brownian movement due to the smaller frictional resistance of the medium. This would be clear from the Equation (9) given above. The diminution in the mean distance between the particles also increases the rate of collisions. We have seen that the adsorption theory does not take these factors into consideration. Recently Smoluchowski has been able to formulate the progress of the coalescence with time.

His attention was drawn to the subject by Zsigmondy. Bredig (*Anorganische Fermente*, 1901, p. 15) suggested as the cause of coalescence an increase in surface tension with a decrease in the electric density on the particles. Zsigmondy modified this idea in the sense that there is an attraction between the particles which increases with decrease in the electric charge. As a result of this attraction he assumes that when one particle comes within a certain distance of another the two coalesce. This distance is taken as a measure of the force of attraction and is called the radius of the sphere of action. It has been shown by Zsigmondy that the time required for a definite colour change in a gold sol gradually decreases with rise in electrolyte concentration till it reaches a minimum,¹ which does not change any further with higher concentrations. This was assumed to prove that the radius of attraction reached a maximum value.

¹ Similar minimum times have been observed with cupric sulphide and mercuric sulphide sols by the writer. A copper sulphide sol gave two minutes as the time necessary for the appearance of visible clots when the concentration of the precipitating electrolyte (barium chloride) was varied from N/300 to N/20. At dilutions higher than N/300 the time was observed to increase as usual. (Mukherjee and Sen, *loc. cit.*)

Smoluchowski utilised this idea of a sphere of action to avoid a consideration of the forces that influence the coalescence. He considers the probability of particles coming within their mutual sphere of action when the radius of the sphere has a constant value determined by the conditions. It is assumed that as soon as a particle comes within the sphere of attraction by virtue of its Brownian movement the two particles coalesce. This discontinuous view of the obviously continuous process of coalescence was assumed to avoid a consideration of the nature and distribution of the forces that are present.

Considering the effect of the motion of each particle and also that each of the aggregates acts as a condensation centre he derives the following equations:—

$$\Sigma n = \frac{n_0}{1 + \frac{t}{T}} \quad (10)$$

$$n_1 = \frac{n_0}{\left(1 + \frac{t}{T}\right)^2} \quad (11)$$

$$n_k = \frac{n_0 (an_0 t)^{k-1}}{(1 + an_0 t)^{k+1}} \quad (12)$$

where " n_0 " denotes the total number of particles originally present per unit volume before coalescence begins. They are all assumed to be spherical and equal in size. " t " is the time in seconds that has elapsed since the electrolyte and the sol have been mixed. " T " is a constant characteristic of the rate of coagulation and is given by

$$T = \frac{1}{4\pi D R_a n_0} \quad (13)$$

where " D " is the diffusion constant as given by Equation (9), $a = 4\pi D R_a$; and R_a is the radius of the sphere of action.

Σn denotes the total number of particles in all stages of coalescence in unit volume when the time is " t "; n_1 denotes the number of the primary particles whose original number was n_0 , at the time " t "; n_k denotes the number of particles of the k^{th} stage of coalescence; that is, the number of aggregates each of which consists of " k " of the primary particles. " k " is evidently an integer. In 1918 Zsigmondy (*loc. cit.*) published the results of an investigation to test this theory. He restricted his investigation to the rate of

decrease in the primary particles (green in the ultramicroscope) in a colloidal gold sol when the minimum time of coagulation has been reached. He found that $Ra = 2.2$ times r , the radius of the particles. Similar values were obtained by Westgren and Reitstötter with more coarsely dispersed gold sols. The value of Ra/r , however, varies in one experiment from 1.4 to 3.8. The recent experiments of Kruyt and Von Arkel show greater variations. They are of opinion that there is some regularity in these variations. They could not observe a maximum value of Ra/r equal to 2. They found a maximum value equal to 0.73.

Smoluchowski, assuming from the data of Zsigmondy available at that time that $Ra/r = 2$, points out that the maximum rate of coagulation is reached when each collision between two particles is successful in bringing about a coalescence. When the rate of coagulation is slower all the collisions are not successful in bringing about a coalescence of the particles. If " ζ " is the fraction of the collisions that are successful in bringing about coalescence then " T " in Equations (10) and (11) takes the form

$$T = \frac{1}{2} \frac{N_0 \cdot \eta}{R \cdot n\lambda \cdot \zeta \cdot \theta} \quad (14)$$

where N_0 , R , θ , $n\lambda$ and η have the same meaning as in Equations (9), (10) and (13).

$$\text{Putting } \frac{1}{2} \frac{N_0 \cdot \eta}{R \cdot \theta \cdot n\lambda} = \frac{1}{\beta} \quad (15)$$

we have

$$\Sigma n = \frac{n_0}{1 + \frac{t}{T}} = \frac{n_0}{1 + \beta \cdot \epsilon \cdot t} \quad (16)$$

Since only " ϵ " is variable a comparison of the coagulation time " t " for the same change in the sol makes it possible to determine the variation in the percentage of successful collisions and its dependence on the conditions of experiment. When the maximum rate is reached $\epsilon=1$ and hence a measure of the absolute value of ϵ is possible.

Problems awaiting Solution.—A glance through the experimental work would show that the assumption of the constancy of " T " is not well justified. The simplicity of Smoluchowski's equations

consists in that there is only one constant. The experimental limitations are great and it is quite possible that the discrepancies are due to the defects of the ultramicroscopic method. The other possibility is that the simplifying assumptions of Smoluchowski, for example, the constancy of "T" independent of the stage of coalescence, are true within narrow limits. It is of great interest to know the limits within which these equations are valid.

The important questions that await solution in this connection are :

- (a) the limits within which the above equations are valid ; and if the above equations are valid,
- (b) the variation of ϵ with concentration of electrolyte ;
- (c) the dependence of ϵ on the electric charge ;
- (d) the variation of ϵ with temperature.

In the following an account of an attempt to examine these factors with the exception of (c) is recorded.

Indirect Methods.—Variations in physical properties that occur simultaneously with the process of coagulation can be utilised to measure the rate of coalescence.

Smoluchowski pointed out that the viscosity measurements of Gann do not satisfy the main requirements of his equations, namely, a similarity in the form of the curves (showing the variation in viscosity with time) independent of the nature of the electrolyte. He concludes that viscosity changes do not form a measure of the coagulation process. Yet he considers that the method is suitable for a quantitative comparison of the effect of various concentrations on the values of ϵ when the curves are similar.

The variation in physical properties, however, is likely to show the validity of the fundamental equations of Smoluchowski. The fact that the curves showing the change in viscosity with time are dissimilar shows that these assumptions are not justified and Smoluchowski thinks that "T" is dependent on the magnitude of the aggregates.

Since as yet it is not possible to express physical properties, e.g., the viscosity or the absorption of light, in terms of definite functions of the number and size of particles, a quantitative comparison of different sols is not possible by indirect methods. We have, therefore, to restrict ourselves to the same sol.

Experiments with Gold Sols.—An examination of the changes in the colour of gold sols on the addition of an electrolyte showed (Mukherjee and Papaconstantinou. T., 117, 1920, 1563) that the

variation in the absorption of light of gold sols afford an easy and accurate method suitable for this purpose. The gold sols prepared by the nucleus method of Zsigmondy conform very nearly to the requirements of Equations (10) to (12) in so far as the particles are fairly uniform in size. It would be very convenient to work with a sol with reproducible properties, as data obtained on different dates with different preparations can be rigorously compared.

The reproducibility of gold sols.—The reproducibility of these sols was, therefore, examined in two ways :

(1) the measurement of the absorption of light by a König-Martens Spectrophotometer corresponding to seven different parts of the spectrum, (2) by determining the time taken for a definite change in the colour corresponding to a definite electrolyte concentration. This time is very sensitive to any change in the condition of the sol.

(1) Twelve samples were prepared under the same conditions and the limits of variation in the extinction co-efficients corresponding to the wave lengths mentioned are given below :—

TABLE XIII.

Wave length in $\mu\mu$	Limits of variation in the Absorption Co-efficients ('K') observed.
683	0.0150—0.0752
602	0.0827—0.1517
583	0.1207—0.1903
563	0.1517—0.2322
547	0.2512—0.2687
523	0.3679—0.4884
506	0.4197—0.4884
475	0.3387—0.3985

The wave lengths mentioned above were determined from the position of the sperometer screw of the instrument with the help of a calibrating curve obtained in the usual way. It will be seen from Fig. 1 that the variation is least in the region $520\mu\mu$ to $540\mu\mu$. The curve is taken from Reprint No. 5 (Mukherjee and Papaconstantinou, T., 1920, 117, 1563-1573).

Fig. 1.

Method of measuring "coagulation times."

(2) The "coagulation times" as defined above show greater variations. They were measured as follows. The gold sols were prepared by the nucleus method of Zsigmondy. The sols contain 0.067 grams of gold per litre. In the cardioid ultramicroscope, the hydrosols show mostly green particles with a few brown ones. The great variation in the colour of a ruby-red gold sol makes it suitable to use a definite shade of violet-red or bluish-violet for comparison. "Standards" for comparison were made by arresting the colour change at a selected stage with gelatine. With care perfectly reproducible standards can be prepared. The "times" necessary for the sol to change to the colour of the standards are given below.

TABLE XIV.

Electrolyte : .852 N/1000 Barium Chloride.

	Violet.		Blue.	
	I	II	I	II
Sol A.	6 min.	6 min.	29 min.	29 min.
Sol B.	2 min.	2 min. 25 sec.	10 min. 45 sec.	10 min. 30 sec.

The results are the mean of six observations taken separately Sol B was obtained by boiling Sol A for a few minutes and cooling. This shows that the boiling produces a change in the sol.

The concentration of the electrolyte used for producing the change in the colour should preferably have a slow rate of precipitation. About 2 c.c. of a 2% gelatine solution (liquid) was added to 10 c. c. of the sol-electrolyte mixture. The same "standards" remain satisfactory for 10 to 12 days. Wide test tubes were used to secure a suitable depth of colour. As is well known, ruby-red gold sols are extremely sensitive to impurities. Reproducible results can only be obtained if the vessels are cleaned with sufficient care. The glass vessels were washed with "conductivity" water after the usual cleaning with hot chromic acid and distilled water. The test tubes were washed with

boiling distilled water (after chromic acid had been used). They were next washed with conductivity water and dried by passing steam derived from conductivity water. There is a noticeable difference between cleaning with distilled water and conductivity water. Table XV illustrates the reproducibility of the results :—

TABLE XV.
N/1000 SR (NO₃)₂. Observations

	1		2		3		4		5		6	
	min.	sec.	min.	sec.	min.	sec.	min.	sec.	min.	sec.	min.	sec.
Violet	1	10	1	6	1	11	1	13	1	0	1	12
Blue	5	45	5	48	5	30	5	30	3	30	6	0

In each case 3-5 readings were taken. With all these precautions, at times discordant results were obtained which are probably due to occasional presence of dust particles. The agreement between the different observations and the appearance of the characteristic blue colour are the surest indications of the absence of impurities. With long intervals it is difficult to avoid dust. Also a slow change in colour is much less sensitive to the eye. For these reasons it was found suitable to use concentrations of electrolytes which change the colour to blue within an hour. As different "standards" were used they are indicated as V₁, V₂, etc., for violet standards and in a similar manner for blue standards.

A sol on standing undergoes somewhat irregular changes which may in part be due to dust particles getting in accidentally. In spite of all precautions one cannot be sure that there is no such variation in a particular sample. This variation is not wholly due to the fungus that grows in these sols. For this reason it is necessary to vary one factor only at a time and compare its effects. Table XVI illustrates the variations. The same "violet" and "blue" standards were used. The times given are the mean of 3-5 observations. The "age" denotes the time that has passed since the preparation of the sol. The sols were kept in resistance glass vessels. Sol I is an old preparation kept for two months. It was boiled twice during this interval to destroy the organic growths.

TABLE XVI.

Electrolyte: .852N/1200 Barium Chloride

Standards.	Sol G.		Sol H.		Sol I		Age.
	min.	sec.	min.	sec.	min.	sec.	
Violet.	3	15	3	35	...		2 days.
"	2	30	8	0	11	0	3 "
"	1	4	6	15	12	30	4 "
"	2	0	6	52	12	30	5 "
"	2	45	4	10	12	30	6 "
Blue.	7	0	6	30	...		2 "
"	5	45	18	0	22	0	3 "
"	4	40	19	0	30	0	4 "
"	9	0	22	30	30	30	5 "
"	9	0	14	0	33	0	6 "

The comparison was therefore restricted to the same sol so long as it showed no variation in its properties.

The Constancy of "T" in Equations (10) to (12) during the Process of Coalescence.

According to the simple assumptions of Smoluchowski the progress of coalescence should be uniformly the same for various electrolytes and for their different concentrations. The constancy of "T" implies that if we assume a series of consecutive stages of coalescence of a sol—under a definite set of conditions, namely, a definite electrolyte concentration and temperature—following each other by intervals of time equal to " δt ," they are each characterised by a definite number and manner of distribution of particles of each category (primary, secondary, etc.). Let us indicate the stage of coalescence corresponding to the time " t " seconds (since the sol and the electrolyte were mixed) under the given conditions by the numbers

$$\Sigma n, n_1, n_2, n_3, \dots n_k \dots \dots \dots;$$

where the subscripts refer to the number of primary particles by the

union of which the aggregate is composed. Thus n_k denotes the number of aggregates, each of which are composed of " k " primary particles. " k " is evidently an integer.

Similarly let us denote the stage of coalescence corresponding to the interval $t + \delta t$ by

$$\Sigma n', n_1', n_2', n_3', \dots n_k' \dots$$

These stages of coalescence are independent of external conditions, so long as Equations (10) to (12) are valid. The only change that external conditions can bring about is a variation in the value of T . That is, if the external conditions are varied the sol will always pass through the same consecutive stages of coalescence, only the rapidity of succession of these stages will be determined by them. Any property which varies continuously with the progress of coalescence without having any maxima or minima can be utilised to characterise the stages of coalescence. For each value of this property is characteristic of the time that has passed since the mixing of electrolyte and sol. According to the equations of Smoluchowski the times taken to reach any particular stage depend only on the value of T which is constant under a definite set of conditions. Let us compare two different electrolytes A and B, of concentrations C_1 and C_2 . Let us suppose that after the time " t ," the stage of coalescence indicated by

$$\Sigma n, n_1, n_2, n_3, \dots n_k \dots$$

has been reached when the electrolyte is 'A' of concentration C_1 . This stage of coalescence has a definite value for the physical property we are considering, and is independent of the value of T . Let us assume that T_1 and T_2 are the corresponding values of T for the two cases. To be definite we shall consider the variation in the total number of particles of all categories, which varies continually with the progress of coalescence. Let us assume that at the times " t_1 " and " t_2 ," both electrolytes have reached a state at which the total number of particles is the same. From Equation (10) we have therefore

$$n = \frac{n_0}{1 + \frac{t_1}{T_1}} = \frac{n_0}{1 + \frac{t_2}{T_2}} \quad (17)$$

$$\text{or} \quad \frac{t_1}{T_1} = \frac{t_2}{T_2} \quad (18)$$

The general equation (12) can be written as

$$n_k = \frac{n_0 \left(\frac{t}{T}\right)^{k-1}}{\left(1 + \frac{t}{T}\right)^{k+1}} \quad (19)$$

Since n_0 and k are constants, if $\frac{t}{T}$ is constant n_k has a fixed value. That is the condition

$$\frac{t_1}{T_1} = \frac{t_2}{T_2} \quad \text{which is deduced from the condition that } \Xi n \text{ has}$$

a fixed value also implies that the values of $n_1, n_2, n_3, \dots, n_k$ are the same in both cases. This means in other words that a definite value of Ξn fixes unequivocally the stage of coalescence. Therefore from the deduction that the successive stages of coalescence are always the same and depend only on the time, any property of the sol that varies continually can be utilised to represent a fixed value of Ξn or n_1 or a definite stage of coalescence. A definite value of this property is thus characteristic of the stage of coalescence. It also follows from the above considerations that all curves showing a variation of this property with time should be similar. A deviation from this similarity, in itself, would mean that the Equations (10) to (12) do not represent the facts.

The absorption co-efficients of gold sols for different wave lengths change on addition of an electrolyte in a complex manner. The theories of the colour of these sols as advanced by Maxwell Garnett (Phil. Trans., 1904, 203, A, 385; 1906, 205, A, 237), and by Mie (Ann. der Phys. [IV], 1908, 25, 377) would lead one to expect that any change in the number and manner of distribution of the particles n_1, n_2 , etc., will produce a great change in the optical properties of the sol. This is in agreement with observations. Now, if the successive stages of coalescence were independent of the nature and concentration of the electrolyte, then the manner of variation of the complex absorption would be the same in each case. The absorption in the red region of the spectrum varies continuously. Corresponding to each value of the absorption co-efficient for a particular wave in this region the values in the other parts should be fixed. If the contrary holds good then the conclusion is obvious that the successive stages of coalescence are not independent of the nature and concentration of the electrolyte as assumed by Smoluchowski.

It has been found that for the stage indicated by the value of the co-efficient of absorption for $683 \mu\mu = 0.4985$ the values of the co-efficient for the other wave length given in the following table in column II are independent of the nature of the electrolyte.

The concentrations of the electrolytes were such as to produce rapid coagulation. In columns III and IV the co-efficients of the "nucleus sol" have been given for the original sol and for the stage of coagulation characterised by the value of the co-efficient for $683 \mu\mu = 0.4156$. Fig. 2 represents the figures graphically and is taken from Reprint No. 5 (Mukherjee and Papaconstantinou, *loc. cit.*)

TABLE XVII.

Wave length in $\mu\mu$.	Absorption Co-efficients (k)			
	I	II	III	IV
683	0.0453	0.4985	0.0376	0.4156
602	0.1055	0.3679	0.1131	0.409
583	0.1518	0.3388	0.1595	0.3936
563	0.2076	0.3294	0.2076	0.336
547	0.2512	0.3236	0.2867	0.3732
523	0.3780	0.3780	0.3780	0.3882
506	0.4647	0.3581	0.3882	0.3780
475	0.3581	0.3198	0.2581	0.3780

Fig. 2.

The changes in colour were observed with a König-Martens spectrophotometer. (Fig. 2.) The changes in the colour from red to blue on the addition of an electrolyte are mainly a result of a change in the absorption-co-efficient of red and violet rays. The change is greatest in the red region. The co-efficients were calculated from the equation (Hildebrand, Zeitzsch, Elektrochem, 1908, 14, 349).

$$K = K_1 / 2.306 = \frac{1}{d} \log_{10} \frac{\tan^2 \alpha^*}{\tan^2 \alpha}$$

* the angle for water or electrolyte solution alone.

In studying the precipitation, equal volumes of sol and electrolyte were mixed. The co-efficients for the pure sol refer to that for the sol diluted with an equal volume of pure water. The electrolytes used were potassium chloride, sodium chloride, barium chloride, potassium nitrate, strontium nitrate, and potassium sulphate.

The results are given in Fig. 2 and are the mean of observations with different samples of sols and different electrolytes. The wavelengths are correct within $\pm\mu\mu$. It is interesting to note that in the region near $523\mu\mu$ there is scarcely any change in the absorption-co-efficient.

The results obtained for the absorption of light by the hydrosols, prepared under exactly similar conditions, differ a little among themselves except in this region, which is close to the spectral region, where the absorption is at a maximum (near $506\mu\mu$) (compare Ehrenhaft, Ann. Physik, 1903, [IV], 11, 489).

Comparison of the values of "T" as a test of Smoluehowski's theory.

Since the absorption co-efficient in the red region varies continuously with the coagulation and its magnitude is sufficiently great, a definite value of the absorption co-efficient for a fixed wave length ($683\mu\mu$) can be taken as representing a definite stage of the coalescence.

In the following tables the absorption-co-efficients at different times are given for the wave length $683\mu\mu$. The tables are taken from Reprint No. 5 (Mukherjee and Papaconstantinou, *loc. cit.*):

TABLE XVIII.

Electrolyte: Potassium Chloride.

Time in minutes after mixing equal volumes of electrolyte and sol.	Absorption co-efficients for various concentrations.		
	N/24	N/26	N/28
.....	0.0453	0.0453	0.0453
0.5	0.3732	0.2867	0.1683
1	0.438	0.3630	0.2257
1.5	0.4497	0.4046
2	0.438	0.2777
3	0.4497	0.3431
5		0.3836
9		0.4263
13		0.438
15		0.4497

TABLE XIX.
Potassium Nitrate.

Time.	Concentration.		
	N/24	N/26	N/30
.....	0.0453	0.0453	0.0453
0.5	0.3336
1	0.4263	0.2866	0.269
1.5	0.4497	0.3271	0.3143
2	0.3629	0.3336
3	0.4156	0.3732
4	0.438	0.394
5	0.4497
8	0.4263
10	0.438
16	0.4497

TABLE XX.
Barium Chloride.

Time.	Concentration.		
	0.852N/900	0.852N/1000	0.852N/1100
...	0.0453	0.0453	0.0453
1	0.2257	...	0.1603
2	0.2867	...	0.2007
4	0.3529	...	0.2687
5	0.3836	0.3051	0.3051
7	0.438	0.3431	0.3237
8	0.4497	...	0.3336
9	0.4497	0.3836	0.3529
11	...	0.4263	0.363
13	...	0.4497	0.363
16	0.3732

The limits within which the rate of coalescence could be varied were restricted by the fact that when the rate is slow the particles begin to settle leaving a clear layer at the top and the measurements are not reliable. Also with time some of the particles stick to the sides of the vessel. Lastly, it is difficult to avoid dust particles for a long time.

The values given in Tables 18 to 20 have been plotted graphically.

Each of these curves is characterised by a definite value of T (or ϵ). Corresponding to the three concentrations of any one of these electrolytes, there are three intervals which must pass in order that the absorption co-efficient may have the same value. These intervals are correlated by the following relation according to equations (10) to (12) or (15).

$$\frac{t_1}{T_1} = \frac{t_2}{T_2} = \frac{t_3}{T_3} \quad (20)$$

$$\text{or} \quad t_1 : t_2 : t_3 = T_1 : T_2 : T_3 = \frac{1}{\epsilon} \quad (21)$$

Since T_1 , T_2 and T_3 are constant the ratio of the time-parameters corresponding to the same absorption co-efficient should be independent of the absolute value of the absorption-co-efficient. Corresponding to different values of the absorption co-efficient we get different values of t_1 , t_2 and t_3 . All these values should show a constant ratio. In the following three tables this comparison is made for the three electrolytes mentioned in tables.

TABLE XXI.
Electrolyte: Potassium Chloride.

Values of absorption co- efficient.	Time in seconds.			Ratios.
	N 24	N 26	N 28	
	t_1	t_2	t_3	$T_1 : T_2 : T_3$
370	27	65	255	1 : 2.47 : 9.44
400	35-40	85	345	1 : 2.3 : 9.3
438	60	120	780	1 : 2.0 : 13.3
445	75	180	900	1 : 2.4 : 12.0
			Average	1 : 2.3 : 11.4
	Extreme deviation from average.			7.4% 19%

TABLE XXII.

Electrolyte Potassium Nitrate.

Absorption Co-efficients.	Times.			T_1 : T_2 : T_3
	$N/24$ t_1	$N/26$ t_2	$N/28$ t_3	
·350	25 } 27 30 }	105	145	1 : 3·9 : 5·4
·400	45	165	255	1 : 3·7 : 5·7
·425	60	210	375	1 : 3·5 : 6·2
·445	90	300	780	1 : 3·3 : 8·6
			Average	1 : 3·6 : 6·3
	Extreme variation from Average			8·8% 33%

TABLE XXIII.

Electrolyte : Barium Chloride.

Absorption Co-efficients.	Times.			T_1 : T_2 : T_3
	t_1	t_2	t_3	
0·300	135	277	345	1 : 2 : 2·55
0·327	165	345	430	1 : 2·1 : 2·6
0·350	225	430	540	1 : 1·91 : 2·4
0·370	265	480	780	1 : 1·81 : 3·0
			Average	1 : 1·93 : 2·64
	Extreme variation from Average			6% 13%

It will be seen from Tables XXI to XXIII that the agreement is as good as can be expected. The variation in T is as great as 11 times but the ratios are constant. The agreement shows that the ratios in the value of T are independent of the time or the stage of coalescence. The ultramicroscopic measurements so far made show even during one experiment a much greater variation in T as will be evident from the following tables :

TABLE XXIII (a).

*Observer—Zsigmondy.*Values of $\beta' = \frac{1}{T}$.

Series D.	Series E.	Series F.
0.083	0.105	0.040
0.028	0.058	0.0195
0.0302	0.049	0.0183
0.0309	0.0475	0.0153
...	0.0403	0.0187
...	...	0.0126

Zsigmondy used high concentrations of electrolyte for securing a rapid rate. When the rate of coagulation is slow and the duration of experiment is greater than a few minutes he found that impossible values of β' are obtained. He thinks that the presence of impurities in the water used in diluting the sol for ultramicroscopic observations, is the cause of this irregularity. In his case the maximum time covered by the experiments is 80 secs. Similarly Westgren and Reitstötter working with coarse gold sols find the following range of variation in the constant.

TABLE XXIII (b).

*Observer : Westgren and Reitstötter.*Values of $\frac{Ra}{r}$

Series I.	Series II.	Series III.	Series IV.
3.74	2.56	2.75	3.41
2.47	2.81	2.60	2.80
2.07	2.33	2.17	2.60
2.10	2.31	2.40	2.48
2.09	2.31	2.12	2.14
1.62
1.41	2.16	2.15	2.15
.....	2.19	2.05
Average 2.2	2.38	2.36	2.19
Extreme			
Variation 75%	10%	17%	55%

Kruyt and Arkel working with Selenium sol and very slow rate of coagulation find extremely wide variations in T in the same experiment.

TABLE XXIV,

Observer : Kruyt and Arkel.

Values of T (in hours).

I	II	III	IV
2.8	260	131	1.3
5.1	390	55	3.4
44	270	52	2.2
(43)	320	54	4.3
(157)	600	68	10.5
200	370	55	40
.....	510
.....	440	48	37
.....	52
.....	38

The above few instances will suffice to show the range of variations in " T " during the course of one experiment; that has been observed in the ultramicroscopic measurements.

Considering that in Tables XXI to XXIII the ratios between the different values of T are taken, the range of variation is extremely small. *The actual deviations in the value of T in any one experiment must be much less than the extreme variations given.* This comparison leaves no room for doubt that " T " is a constant in the case of gold sols and within the limits of the rate of coagulation that have been studied. In fact these data constitute the best evidence so far recorded in favour of the theory of Smoluchowski.

The Dependence of ϵ on the Concentration.

Tables 21 to 23 show clearly how rapidly ϵ the percentage of successful collisions increases with concentration. A change of concentration in the ratio 24 to 28 increases the rate in the ratio

1 : 11 or 1 : 6 as the case may be. It would be extremely interesting to work with a sol which is less susceptible to impurities than these gold sols.

Variation of T or ϵ with Temperature.

Similarly by determining the times required to produce a definite change in the colour of the sol for the same electrolyte concentration but different temperatures we can determine the variation in ϵ with temperature.

From equation (12)

$$\Sigma n = \frac{n_0}{1 + \beta \cdot \epsilon \cdot t}$$

$$\text{we get } \frac{n_0}{\Sigma n} = 1 + \beta \cdot \epsilon \cdot t \quad (22)$$

Since a definite change of colour is being used $\frac{n_0}{n}$ is constant or

$$1 + \beta \cdot \epsilon \cdot t = k_1 \text{ a constant} \quad (23)$$

Substituting the value of β in (15) we get

$$1 + \frac{4R \cdot \theta \cdot \eta_0}{3N_0 \cdot \eta} \cdot \epsilon \cdot t = k_1 \quad (24)$$

Since R , η_0 , and N_0 are constants we have

$$\frac{t \cdot \theta \cdot \epsilon}{\eta} = k_1 \text{ a constant} \quad (25)$$

The viscosity of colloidal gold solutions has been found to be practically equal to that of water and the variation with temperature can be assumed to be equal to that of water. For different temperatures we have

$$\frac{t_1 \cdot \theta_1 \cdot \epsilon_1}{\eta_1} = \frac{t_2 \cdot \theta_2 \cdot \epsilon_2}{\eta_2} \quad (26)$$

Since t is experimentally determined and θ and η are known variations in ϵ can be compared.

The experimental data are given below. They are taken from Reprint No. 5, pp. 1570-1.

TABLE XXV.

Electrolyte.	Standards.	Temperatures.		
		15°	30°	50°
N/30 potassium chloride ...	V Sol. C	5 min.	10 min.	8 min. 30 sec.
N/30 potassium sulphate ...	" " D	30 sec.	10 sec.	10 sec.
N/30 potassium nitrate ..	" " D	42 sec.	18 sec.	12 sec.

TABLE XXVI.

Electrolyte : Barium Chloride. Sol. E.

Concentration.	Standards.	Temperatures.			
		15°	30°	40°	50°
0·852N/1000		7 min.	6 min.	4 min. 50 sec.	4 min. 20 sec.
0·852N/1000		34 "	23 "	...	13 " 30 "
0·852N/1200		23 "	13 "	12 min. 30 sec.	6 min. 15 "
0·852N/1200		124 "	74 "	62 min.	...

TABLE XXVII.

Electrolyte : Strontium Nitrate. Sol. F.

Concentration.	Standards.	Temperatures.		
		15°	30°	50°
N/1000		1 min. 10 sec.	20 sec.	8 sec.
N/1000		8 " 15 "	1 min. 40 sec.	45 "

At 15°, 30°, 40°, and 50° η/θ has the values $3\cdot96 \times 10^{-5}$, $3\cdot96 \times 10^{-5}$, $3\cdot31 \times 10^{-5}$, $2\cdot1 \times 10^{-5}$, and $1\cdot7 \times 10^{-5}$ respectively. The values for the viscosity are taken from the tables in Kaye and Laby's book on Physical and Chemical constants, P. 30, 1919.

From Equation (26) we have

$$\epsilon_{15^{\circ}}:\epsilon_{30^{\circ}}:\epsilon_{40^{\circ}}:\epsilon_{50^{\circ}}=(\eta/t\theta)_{15^{\circ}}:(\eta/t\theta)_{30^{\circ}}:(\eta/t\theta)_{40^{\circ}}:(\eta/t\theta)_{50^{\circ}}$$

TABLE XXVIII.

Electrolyte.	...	Temperatures		
		15°	30°.	50°
N/30 Kcl ...	$\eta/t\theta \times 10^7$	1.32	0.50	0.33
N/30 K ₂ SO ₄ ...	"	13.2	33	17
N/30 KNO ₃ ...	"	9.43	18.4	14.0

TABLE XXIX.

Electrolyte : Barium Chloride.

Concentration.	Standards.	Temperatures.			
		15°	30°	40°	50°
·852N/1000 ...	V ₂ $\eta/t\theta \times 10^5$	9.43	9.2	7.24	6.54
" ...	B ₂ "	1.94	2.4	...	2.1
Ratio between ...	" $\eta/t\theta$ " } V ₂	100 :	98 :	77 :	70
" " ...	" } B ₂	100 :	123 :	...	108
·852N/1200 ...	V ₂ $\eta/t\theta \times 10^5$	2.87	4.24	2.8	4.5
" ...	B ₂ "	53.22	74.5	64	...
Ratio between ...	$\eta/t\theta$ } V ₂	100 :	148 :	98 :	158
" " ...	" } B ₂	100 :	140 :	106 :	"

TABLE XXX.

Electrolyte : Strontium Nitrate.

Concentration.	Standards.	Temperatures.		
		15°	30°	50°
N/100J	V ₃ $\eta/t\theta \times 10^6$	0.565	1.65	2.1
"	B ₃ " $\times 10^7$	0.80	3.3	3.77
Ratio between } ...	V ₃	100 :	291 :	371
$\eta/t\theta$ } ...	B ₃	100 :	410 :	430

Since $\eta/t\theta$ is a constant for a definite electrolyte concentration and temperature according to Smoluchowski's equation the ratios should be independent of the standard used. This is true within the limits of experimental error with .852N/1200 Barium Chloride. In the other two cases the variations are not great considering that we are comparing the ratios. A slight variation in each value will be magnified in the ratio. Taking into account the probable experimental error it can be said that ϵ is roughly constant in each experiment.

On the other hand the variation in ϵ with temperature is considerable. We have already seen that the irregularity in the variation of ϵ means that the precipitating power of the ions change with the temperature.

PART VII.

The Measurement of the Electrical Charge of Colloidal Particles by the U-tube Method.

The stability of a sol is dependent on the electric charge of its particles and it is of interest to know how the charge varies under different conditions. Thus, it is desirable to determine the variation of the percentage of successful collisions ϵ with the potential of the double layer.

[An improved method free from the sources of error inherent in the U-tube method as it is used at present has been devised. This portion of the thesis has been published in the Proceedings of the Royal Society, A, Vol. 103, (1923), pp. 102-121.]

PART VIII.

Discussion.

It is generally admitted that the stability of the so-called suspensoids is due to the electrical charge of the particles. Some of these suspensoids can be kept indefinitely without any separation of the colloid. The numerous collisions that occur between the particles during this time do not, therefore, bring about their stable coalescence. On the other hand, in most sols the colloid separates slowly with time. Thus we see that with pure sols the protective action of the charge can be sufficient or insufficient to overcome the cohesive forces that bring about the coalescence. The stability of the sol is determined by the relative magnitudes of these forces.

As to the manner in which the electric charge prevents the coalescence, nothing definite is known. It is suggested by some that before the two colliding particles can come in contact the double layer must be disrupted. Hatschek (Koll. Zeitsch., 1911, 9, 159) suggested that the stability is due to repulsion between the similar charges on the particles. Donnan pointed out that (Trans. Far. Soc., 1913, 9, P. 20, 28, c.p. also Smoluchowski, *loc. cit.*) it is not quite justifiable to treat the electrical forces between two colloidal particles when they come near each other as to be due simply to the repulsion between two like charges. The presence of an electrical double layer introduces complications. There is, however, some evidence of the existence of repulsive forces. Perrin and Constantin (C. R. 1914, 158, 1168, 1171, 1341) found that repulsive forces should be assumed to exist when the number of particles per unit volume is large. It is possible that when two colloidal particles approach each other closely the electrical charges in the double layer become differently distributed. The consequent difference in electrical energy will manifest itself as a repulsive force as it involves the motion of electrical charges. The attractive and repulsive forces between the particles in some instances produce a regular orientation of the particles. The experiments of Freundlich on the double refraction of vanadium pentoxide sols can be mentioned in this connection.

Anything that would decrease the density of the electrical charge will favour the action of the cohesive forces. Hardy suggested that coagulation becomes instantaneous when the iso-electric point is reached. The experiments of Ellis and Powis have shown that coagulation can take place before the iso-electric point is reached. The narrow limits of the concentrations which produce instantaneous coagulation on one hand and no sensible change in the sol on the other hand, have led Powis (*loc. cit.* c.p. also Zsigmondy) to postulate a critical potential below which coagulation is possible. The experiments on the influence of the quality of the sol, of the dilution and of the hydrogen sulphide content, recorded here show that it is not possible to characterise any colloid by ascribing to it a definite critical potential determining its stability. What seems to be more plausible is that depending on the potential of the double layer a certain amount of work has to be done in order that two particles can come sufficiently near each other to bring into play the forces of cohesion. Only those particles which have sufficient kinetic energy to do this

work can approach each other sufficiently near. The union can be stable only when the attractive forces are greater than the opposing electric forces that the union may result in. The energy term representing the energy which must be spent to do work against the electric forces, may be taken to determine the percentage of successful collisions in Smoluchowski's equations (*loc. cit.*). Assuming Maxwell's distribution law to hold good the fraction of particles having kinetic energy greater than A is given by $e^{-A/kT}$ where k is given by $R/\text{No.}$, R is the gas constant, No. , the Avogadro number and T the absolute temperature. " ϵ " is thus proportional to $e^{-A/kT}$. The exponential form of the expression explains the rapid variation in " ϵ " with the concentration of an electrolyte. The electrolyte concentration and hence the potential of the double layer determined the rate of coagulation that is being considered. A , the work against electric forces, diminishes as the electric charge diminishes.

For a complete understanding of the process of coagulation it remains to explain the electrical charge on the nature and the concentration of the electrolyte and to define the nature of the forces that bring about the coalescence of the particles.

The diminution in the electric charge on the addition of an electrolyte is referred to the adsorption of ions. Regarding the nature of this adsorption nothing definite is known except that the adsorption-isotherm of Freundlich has been shown to express the equilibrium conditions in a number of cases. In the second portion of this thesis this question is dealt with under the heading, "the Origin of the Charge of Colloidal Particles and its Neutralisation by Electrolytes." The roll of electrostatic forces has been considered.

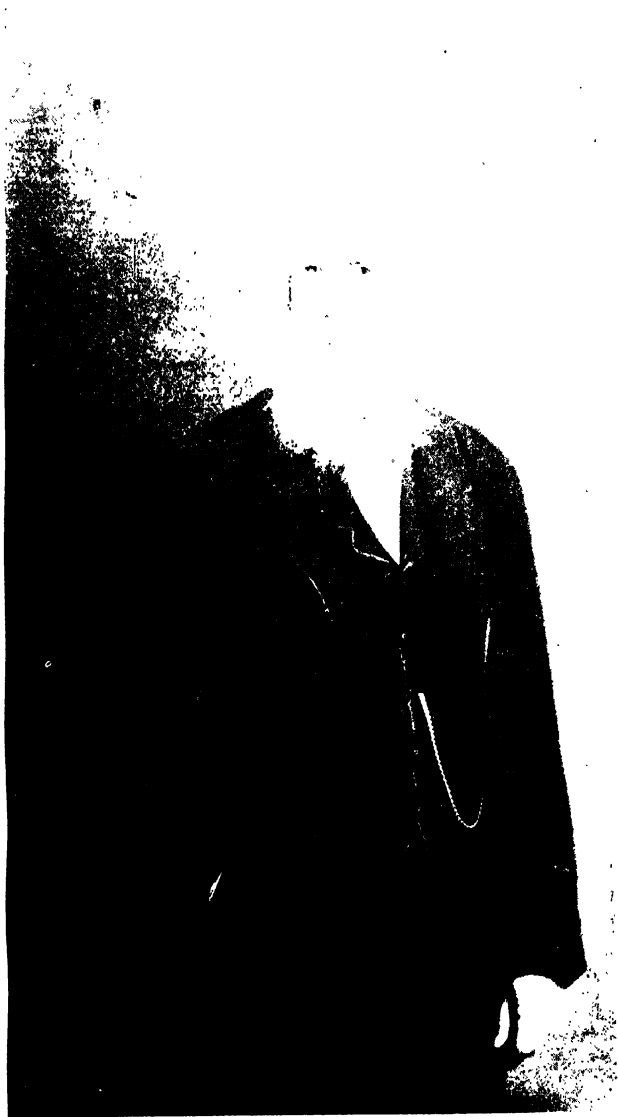
As to the nature of the forces that bring in coalescence, they seem to be chemical in nature. According to Zsigmondy the coalescence does not lead to any such fusion as when two drops coalesce to form a bigger drop. He pointed out that a consideration of the diminution in the total surface (surface energy) is not capable of explaining the coalescence. There is some evidence to show that the surface of the particles do not always touch each other but are rather separated by a layer of water molecules. It is not possible to say anything definitely about the nature of these forces. A general consideration of these forces is, however, necessary to elucidate some facts connected with the process of precipitation. The action of protective colloids can be

attributed to a modification of these forces for the change in the electrical charge is not capable of explaining the stability in a consistent manner. It is possible to imagine that the adsorbed ion also affects the surface forces. Such an assumption seems necessary to explain the observation of Powis (*loc. cit.*) that an arsenious sulphide sol has different electric charges at the point of instantaneous coagulation for potassium chloride and barium chloride respectively. In this case the equations of Smoluchowski are not expected to be valid in the sense that the manner of coalescence depends on the nature of the electrolyte. Similarly, this possible effect of the adsorbed ions on the cohesive forces at the surface may be the cause of the dissimilar forms of the viscosity curves obtained by Gann (*loc. cit.*).

[Thesis presented for the Degree of Doctor of Science, University of London—May 1921.]

AGRICULTURE

Science, VI.



SIR JOHN BENNET LAWES, BART.

Founder of Rothamsted Experimental Station

Agricultural Research and Practice in Europe.

I

The Rothamsted Experimental Station, Harpenden, Herts, England.

By

N. N. GANGULEE,

Khaira Professor of Agriculture, Calcutta University.

The large boulder of granite in the front of the the Rothamsted Laboratory, bearing the following inscription, "To commemorate the completion of fifty years of continuous experiments (the first of their kind) in agriculture, conducted at Rothamsted by Sir John Bennett Lawes and Joseph Henry Gilbert, A.D. MDCCCXCVII," represents a national expression of the recognition of the services rendered by the two pioneers of scientific agriculture; but, the Rothamsted fields, set up by them, where a series of experiments have been carried out since 1843, stand as a place of pilgrimage for all agricultural investigators. It is perhaps, with the single exception of Boussingault's station at Bechelbronn in Alsace, the first agricultural experiment station in the world. The German station at Möckern dates back to 1852, and that in America at Middletown, Connecticut, to 1875.

Though living in the midst of an industrial era, the gifted owner of the Manor of Rothamsted had the foresight to realise that the prosperity of a nation depends, to a large extent, on the development of agriculture. What he undertook as a private citizen, has now become a National Institution, and Great Britain is proud of possessing the oldest and the most unique agricultural research station of the world. I am tempted to lay much stress on the great undertaking of Sir John Lawes, which he carried on exclusively at his own expense in his estate; for his example may be a source of inspiration to those of our landlords who desire to improve the agricultural conditions of their estate, but look to the *State* for inaugurating enterprises calculated to benefit agriculture. The State has become a sort of fetish to us, and we keep our gaze steadfastly fixed on the State-policies, always anticipating that the State authorities would do all for us. Whatever argument, historical and psychological, there may be in the explanation of such a lack of self-reliance among the people themselves, it can be asserted that a great advance may be made in matters of agricultural development, and also revitalizing the rural life of India, if some of our rich landowners seriously undertake agricultural enterprises in their own estates.

Anyone who cares to look into the history of the growth of the Rothamsted Experimental Station would realise the truth of the aphorism "Great undertakings have small beginnings." Out of the series of pot experiments, started as early as 1834, to investigate solubility of phosphatic manures and their application to root-crops, there grew extensive field plots covering to-day an area of about 300 acres; and the genesis of the well-equipped chemical laboratory may be traced to a bedroom fitted up for chemical research by young

Mr. John Lawes soon after his return from Oxford. In his autobiographical note¹ he refers to it as follows:—

“At the age of twenty, I gave an order to a London firm to fit up a complete laboratory, and I am afraid it sadly disturbed the peace of mind of my mother to see one of the best bedrooms in the house fitted up with stoves, retorts, and all the apparatus and reagents necessary for chemical research.”

With the assistance of a young chemist named Dobson, the work was started; but, later on when the field trials were undertaken, Dr. John H. Gilbert, a distinguished English chemist, was appointed. The historical importance of the Rothamsted experiments may be realised from the fact that a series of field experiments conducted by Boussingault on his farm at Bechelbronn in Alsace, though the first of their kind, were unfortunately discontinued while the Rothamsted fields are still going on since 1843. To my mind, the meeting of these two pioneers, Lawes and Gilbert, is one of those remarkable incidents in which we cannot but see the hand of Providence. Perhaps without the the co-operation and meticulous care of Gilbert, Lawes would not have had the success in making such a valuable contribution to our knowledge of agricultural chemistry. They lived and worked together for a period of nearly 60 years. Lawes died in 1900, and Gilbert in 1901.

Sir John Lawes left an endowment fund of £100,000 to the Lawes Agricultural Trust for the maintenance of the work, but as the income from the Trust Fund was not sufficient for the expansion of the research work, a Society was formed for the purpose of providing funds. The development and reorganisation of the Station began

¹ A letter to the late Mr. John Chambers Morton, Editor of “the Agricultural Gazette” in 1888.

soon after the appointment of Mr. A. D. Hall (now Sir Daniel) as the Director. The interesting fact about its expansion is that State-aid was preceded by donations from private sources. In 1906, Mr. J. F. Mason, M. P., presented the Trust with £1,000 for the building and equipment of the Bacteriological Laboratory together with a grant towards its maintenance. In 1907 the Goldsmiths' Company made a grant of £10,000, and it was then possible to organise a department exclusively for soil investigations. A grant of £2,000 came from the Permanent Nitrate Committee.

The State recognition of the Rothamsted Experimental Station as the Research Institute for soil and plant nutrition problems was followed by a grant from the Development Commissioners in 1911, and substantial help for further extension of buildings came from the Ministry of Agriculture after the cessation of the war.

The progress of the research work and the reorganisation of the Station are mainly due to the untiring efforts of the present Director, Sir John Russell. The scientific and administrative staffs increased from 24 persons in 1912 to 67 in 1923. In comparison with what I saw in 1911, there has been expansion in various directions, both in equipment of the farm as well as the laboratories. The departments of plant pathology, protozoology and statistics are among the recent additions to the Station. Since the war, agriculturists have realised the importance of improving the standard of British farming, and therefore efforts are being made by the Station to disseminate the knowledge of scientific agriculture among the farmers of the country, and co-operation with schools, farmers' clubs and similar organisations is also encouraged. On an average 1,300 persons visit the Station per annum.

Most of the Rothamsted work is concerned with the fundamental problems related to the production of crops.

The reliability of the investigations carried on here may be ascribed to the method of combining field observations and data with that of laboratory work. In order to be able to form an accurate idea of the nature of the work, it is necessary to divide this account into two main groups:—Laboratory investigations and those that are conducted in the famous Rothamsted fields.

1. Laboratory Investigations.

(Department of Chemistry.)

To understand Lawes' and Gilbert's work, it is necessary for the student to refer to Baron Liebig's report¹ to the British Association, in which he maintained what is known as the "Mineral theory" of plant nutrition. But the importance of mineral plant food had long been established by De Saussure's experiments² showing that the ash constituents are derived from the soil, and that, though the atmosphere is the source of the greater part of plant food, the minerals found in plant ash are indeed essential to its growth. Liebig went a step further and was so much obsessed with his theory—the ammonia of the air being the source of nitrogen for plants—that he asserted "if the soil be suitable, if it contains a sufficient quantity of alkalies, phosphates and sulphates, nothing will be wanting. The plants will derive their ammonia from the atmosphere as they do carbonic acid."³ With regard to nitrogen supply he definitely states that "the soil cannot be exhausted by the exportation of products containing nitrogen (unless these products contain at the same time a large amount of mineral

¹ Liebig, Justus. "Chemistry in its Application to Agriculture and Physiology," 1847 (4th Ed.).

² De Saussure, Theodore. "Recherches chimiques sur la végétation," 1804, Paris.

³ Liebig, Justus. *Farmers' Magazine* (1847), Vol. XVI, p. 511.

ingredients), because the nitrogen of vegetation is furnished by the atmosphere and not by soil. Hence also we cannot augment the fertility of our fields or their powers of production, by supplying them with manures rich in nitrogen, or with ammoniacal salts alone. The crops on a field diminish or increase in exact proportion to the diminution or increase of the mineral substances conveyed to it in manure."

While a storm of controversies raged over Liebig's nitrogen theory, Lawes and Gilbert planned out a series of experiments to investigate the source of nitrogen for plants. By 1885, they were able to publish several papers¹ with the following conclusions²:—

(a) Compositions of ash constituents are not the reliable criterion as to the plant needs.

(b) An adequate supply of some nitrogenous compounds, such as nitrates and ammonium salts, are necessary in order to obtain increased plant growth. Therefore the application of artificial manures assists materially in the maintenance of soil fertility.

(c) "The beneficial effect of fallowing lies in the increase brought about in the available nitrogen compounds in the soil."

The meticulous care and scientific accuracy with which the investigations were carried out left no room for any controversies, and thus one of the fundamental facts of plant nutrition was established by Lawes and Gilbert. From the collection of bottles of samples arranged in a capacious "Sample House," one may realise the vast amount of soil-sampling and analytical work done since the beginning of the Rothamsted experiments. The total

¹ Lawes and Gilbert. "On the composition of the ash of wheat grain and wheat straw grown at Rothamsted in different seasons and by different manures." *Trans. Chem. Soc.*, 1884, XLV, 305-307.

² Russell, E. J. "Soil Conditions and Plant Growth."

number of samples of soils, seeds and plant ashes would now probably exceed 100,000 bottles.

But what actually happens by the application of nitrogenous manures to soils and in what form they are assimilated by plants still remains shrouded in mystery. The importance of nitrates to plant growth has been recognised, but how nitrates are produced in the soil still has to be investigated. Till 1877 it was thought the reaction involved was a chemical one, when Schloesing and Muntz¹ announced that the process of nitrification was due to micro-organic soil population, or to quote their own expression, "organised ferments." Warington² then working in the Rothamsted Station, undertook a series of careful experiments to study the details of the processes involved. He found that nitrification takes place in two successive stages, namely, conversion of ammonia into nitrites, and nitrites into nitrates. Warington's experiments indicating that nitrogenous compounds, in whatever forms they may be, rapidly change to nitrates in the soil and the accumulated data from the Rothamsted fields showing the effect of such compounds on the plant growth emphasized the utmost importance of nitrates to crops as being their source of nitrogen. But the agencies responsible for nitrification have not yet been discovered and it was left to Winogradsky³ to identify specific organisms now known as nitrifying bacteria. His classical research established that (a) the organisms obtain their energy

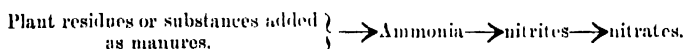
¹ Schloesing and Muntz. "Sur la nitrification par les ferments organisés," *Comp. Rend.*, 1877, LXXXIV, 301-303; (1877) LXXXV, 1018-20.

² Warington, R. On Nitrification, Part I, *Journ. Chem. Soc.*, 1818, XXXIII, 44-51; Part II, *ibid*, 1879, XXXV, 429-55; III, *ibid*, 1884, XLV, 647-72; Part IV, *ibid*, 1891, LIX, 484-529.

³ Winogradsky, S. "Recherches sur les organismes de la nitrification." *Ann. de l'Inst. Pasteur*, 1890, IV, 1' *Memoire*, 213-31; 2' *Memoire*, 257-75; 3' *Memoire*, 760-71.

from oxidation of ammonia, and that (b) there exists a definite relationship between the amount of oxidised product of ammonia and the amount of carbon assimilated.

It follows, then, that the soil must contain ammonia in order to effect the first stage of nitrification. But, as not more than a trace of ammonia could be found in arable soil, the only logical conclusion is that the process of oxidation goes on more vigorously than the rate at which ammonia is formed. Again, the products of this first stage, nitrites, are oxidised at a greater rapidity than the rate of formation. The reactions referred to above, known as ammonification and nitrification, are by no means simple; but as now most of the fundamental problems of agricultural chemistry centre round the complexity of the conversion of the nitrogenous compounds in the soil, it is necessary to have a clear conception of the existing knowledge of these reactions. The chain of ammonification and nitrification processes may be drawn as follows:—



But the complex nature of the reaction arises from several factors which bring into play a series of chemical and bio-chemical changes in the soil. There is the so-called denitrification process which reduces nitrates through certain bacteria in the presence of organic matter under anaerobic conditions. Then there is the process of oxidation of numerous complex plant substances evolving carbon dioxide which is probably the source of energy for micro-organic life. Cellulose, which forms the chief constituent of the plant residues, is also supposed to be decomposed through bacterial agencies. To impress on the reader the fundamental relationship between different processes evolved in the soil, I reproduce

I.

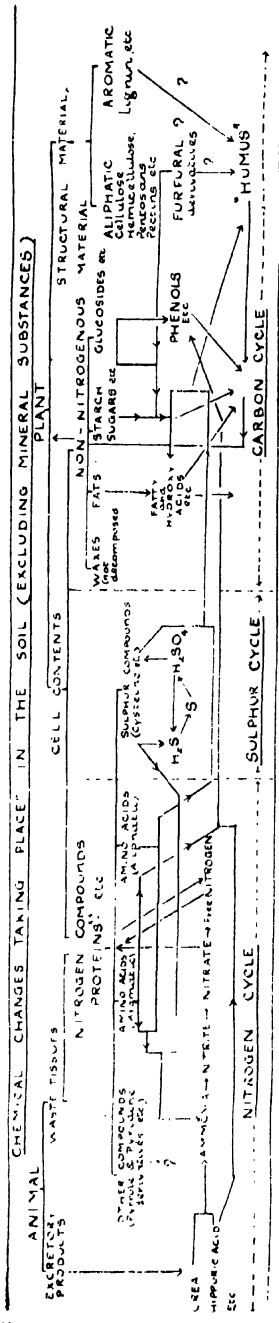


FIG. 1

here the excellent diagram drawn by the Department of Chemistry, the careful study of which will throw much light on the present-day problems of agricultural research.

(See diagram 1.)

Since the foundation of the Rothamsted Station, the nitrogen problem has been one of the chief items of chemical investigations. Nitrogen of soils and crops from the field plots as well as from pot culture experiments is determined, and as high as 1,500 nitrogen determinations are made every year using the well-known Kjeldahl method. The possibility of improving analytical technique is also being considered by Page, the Head of the Department, and his assistants.

The chemical investigations of both the nitrogen and the carbon cycle are now in progress; but the analytical methods followed hitherto for the study of the complex nitrogenous compounds of the soil require to be developed further, before our knowledge of the actual changes occurring in the soil may be advanced. Matthews¹ working in this laboratory, devised an improved method for determining ammonia in the soil, and as in the process adopted, it is possible "to recover 99.5% of added ammonia as against a recovery of 50-60% by the older methods," the details of the apparatus may interest our students of Chemistry.

The apparatus chiefly consists of an aeration tube made of strong glass, 83 cm. long, 2-3 cm. in diameter with a bulb above, and of an absorption vessel, whose height is 40 cm. and the internal diameter 1.8 cm. 25 grams of sieved soil (3 mm. sieve) are introduced into the aeration tube and 50 c. c. of an alkaline solution

¹ Matthews, D. "The Determination of Ammonia in Soil," Jour. Agric. Science, Vol. X (1920), pp. 72-85.

containing 108 grams of sodium carbonate crystals and 150 grams of sodium chloride per litre are added together with about 1 c. c. of paraffin oil. The absorption vessel contains from 5 to 10 c. c. of $\frac{N}{50}$ sulphuric acid diluted with water, and a few drops of a 0.05 per cent. solution of Xethyl red.

(See diagram 2.)

In Diagram 2 A represents the aeration tube and F the absorption vessel. The air enters through B (diameter about 4 mm.) and passes through the rubber stopper in the lower end of the aeration tube. B is supported in a groove cut in the cork block C which is attached to the aeration tube. The upper end of the aeration tube is closed by a rubber stopper carrying a funnel, D, which is loosely plugged with cotton wool to stop spray. D is connected with F by means of a rubber stopper and bent tube E. H is a splash bulb fixed in the stopper, and the arrangement indicated by J is to prevent splashing. The side tube of F is supported by a piece of thick-walled rubber tubing G, split lengthwise.

"The diagram shows only one set of apparatus, but six may be used, the air passing through in turn," says the author. The dotted lines show the structure of two wooden frames of a suitable stand that may be constructed to hold six apparatuses. K represents four cross bars, about 92 cm. long and M wedges of cork or wood to prevent the aeration tubes from slipping down.

The passage of the air current for three hours is considered sufficient in ordinary cases, although by aerating for six hours, as high as 98.5 to 99.5% of the total ammonia can be recovered from soil. It must be noted that the complete recovery of ammonia from soil containing chalk (known as calcareous soil) is difficult, unless it is finely ground, which is not possible in the case of wet

I.

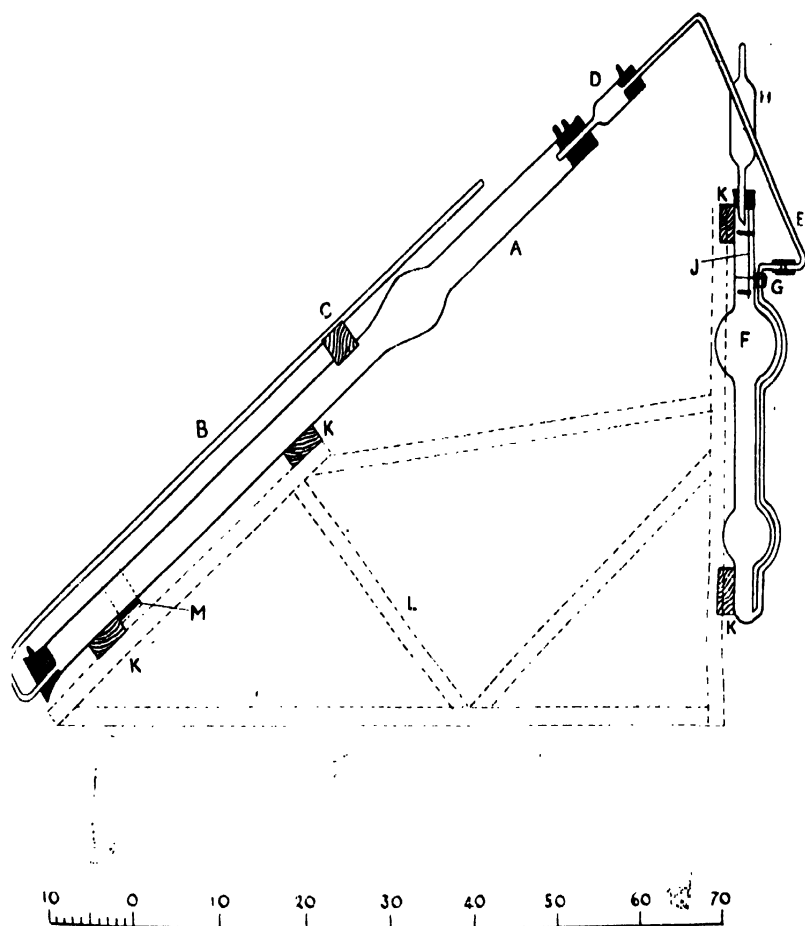


FIG. 2

I.

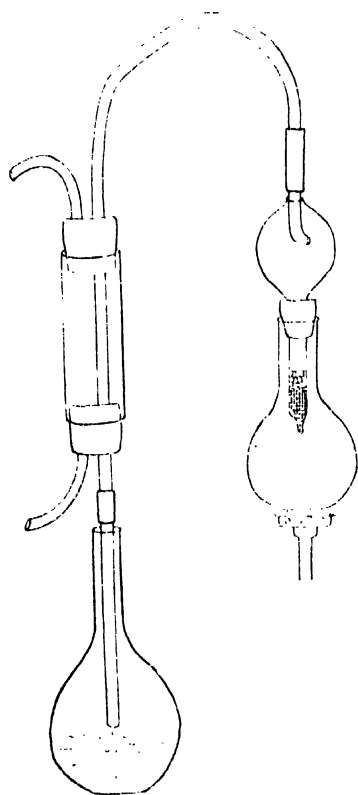


FIG. 3

I.

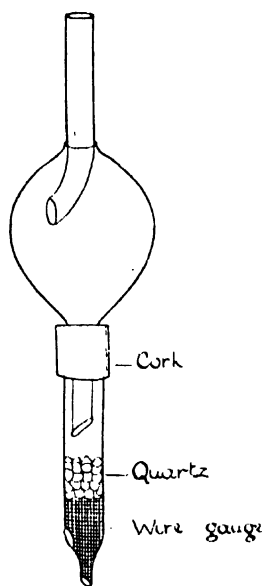


FIG. 4

soils. However, satisfactory results obtained by the method in most cases have made possible further investigations with regard to changes that occur in the process of decomposition of plant residues or green manuring crops ploughed in, or in the application of ammoniacal fertilizers.

I must also refer to the modification worked out by the Department in the method of estimating *nitrate* nitrogen. In our laboratory we use the phenol sulphonic acid method, wherein the possibility of errors in nitrate estimation is not quite eliminated. The method worked out at Rothamsted is as follows:—

Put 200 grams of soil, dried at $50^{\circ} - 55^{\circ}\text{C}$, in a Buchner funnel fitted to a filter flask, and pour on hot distilled water. To an aliquot part of this filtrate, add 10 c. c. of $\frac{2}{N}$ NaOH and 10 c. c. of 3% KMnO_4 . After being digested for six hours, 5 c. c. of alcohol, 100 c. c. of $\frac{2}{N}$ NaOH, and 3 grams of Devarda Alloy (50% Al, 45% Cu and 5% Zn) are added to the solution, and the mixture is distilled for an hour or so.

(See diagram 3.)

The titration is carried out in the usual way using $\frac{N}{50}$ acid, and Methyl red as indicator. Care should be taken to boil off CO_2 before titration. With regard to distillation, great precaution has to be taken to prevent the carrying over of alkali by the fine spray caused by the evolution of hydrogen.

(See diagram 4.) .

A survey of the total nitrogen content of the soil of two of the classical Rothamsted fields where wheat is being grown since 1843 and barley since 1852, is another

interesting piece of work the Department has taken in hand. I shall refer to this nitrogen question in my account of the field experiments.

In Diagram No. 1, we find that Humus is one of the resultant products from the decomposition of plant residues, but the evidences with regard to the processes involved as well as to the formation and composition of humus and similar complex substances are not conclusive. Therefore, the most promising line of research being investigated by Page and Du Toit is the study of humic products—their properties, fractionation—and also of the mechanism of the process by which carbohydrates or similar substances are decomposed into humus.

Perhaps a few introductory remarks as to humic products may be necessary in order to form a clear conception of the research referred to above.

It has been found that amino-acids and pentose compounds are formed when organic substances of soils are allowed to decompose, and the reactions set in by the decompositions are extremely complex. Beckley¹ suggests after a preliminary investigation at Rothamsted that the formation of humus in the soil proceeds in two stages: Carbohydrates, such as cellulose bodies, react with amino-acids producing hydroxymethyl-furfural which condenses to form humus. As further evidences have to be adduced for the acceptance of his suggestion, Page and Du Toit are working out physico-chemical methods for the examination of humus fractions. Recent researches indicate that there may be other sources of humus, such as lignins, phenolic substances, oxidized product of quinones, etc. The lignin which constitutes an important material in the plant structure is now supposed to be the probable mother substance of

¹ Beckley, V. A. "The Preparation and Fractionation of Humic Acid," *Jour. Agric. Science*, Vol. XI, 1921, pp. 66-77.

humus. The problem is enmeshed with innumerable complex reactions, both physico-chemical and microbiological, and suitable laboratory technique has to be developed before any conclusive results are expected from the researches.

With regard to the exact constitution of humus, the position is no better. Its role in soil fertility has been supposed to depend largely on the amount of soluble humus present in the soil; and, apart from being a source of plant food, its presence in the soil-constituents greatly modifies the physical properties of the soil by increasing its water-holding capacity, thus imparting conditions of good tilth and aeration. From the standpoint of Indian agriculture, the problem of supplying organic constituents of soils is vitally important; and therefore, the investigation of the processes involved in the decomposition of organic substances should be undertaken in India, where the climatic conditions are decidedly favourable for such work. Unfortunately, the problem of decomposition of organic substances in soils has not yet been studied in any of the Agricultural Institutes in India, and it may now be hoped that the well-equipped Physico-chemical Laboratory of the University College of Science will find some interest in it.

I must now pass on to other aspects of chemical investigations, that is, the determination of manurial values of phosphatic and potash fertilizers.

The nucleus of every living cell in plants and animals is known to contain phosphorus. The ash contents of plants contain a certain percentage of phosphatic compound, the fact which led De Saussure to accept the findings of Lord Dundonald¹ that the value of bones was due to phosphoric acid contained in the manure.

¹ Lord Dundonald. "Treatise on the Connection of Agriculture with Chemistry," 1795.

According to Hopkins, one ton of wheat bran contains about 24 pounds of phosphorus or 1.22 per cent.¹

In England bones were used to a certain extent as a fertilizer long before 1840, the supply of which was chiefly drawn from North Europe. Soon after Liebig's suggestion of dissolving bones by acid, England took the lead in the manufacture of artificial manures from bones and mineral phosphates. Lawes treated ground coprolites with sulphuric acid and obtained the most useful fertilizer known as superphosphate in the factory at Deptford, and popularized its use through the results of the Rothamsted field experiments. Referring to the extensive use of bone manures in England, the famous German chemist, Liebig, wrote as follows :

"England is robbing all other countries of their fertility. Already in her eagerness for bones, she has turned up the battlefields of Leipzig and Waterloo and of the Crimea; already from the catacombs of Sicily she has carried away the skeletons of many successive generations. Annually she removes from the shores of other countries to her own the manurial equivalent of three million and a half of men, whilst she squanders the contents of her sewers discharging them into the sea. Like a vampire she hangs on the neck of Europe, nay of the whole world, and sucks the heart-blood from nations without a thought of justice towards them, with a shadow of lasting advantage to herself!"²

The progress of arable farming in England from the beginning of the nineteenth century till 1880, characterised by the increased crop-production, must be the underlying reason of such an outburst from a German scientist.

¹ Hopkins, C. G. "Soil Fertility and Permanent Agriculture," Ginn & Co. Boston, 1911.

² *Vide* Baron Liebig's Works (1847).

With the development of scientific research in relation to agriculture, the use of artificial manures was greatly extended in Germany. She utilises in agriculture a very large amount of basic slag, a waste-product obtained from steel manufacturing process, although the credit of discovering its fertilising value must be given to the British investigators. Our present knowledge as regards the effectiveness of basic slag as a phosphatic manure is derived from the classic experiments at Cockle Park carried out for over 25 years. But unfortunately, now that its use has become popular among farmers (the requirement of basic slag in Great Britain is estimated by Middleton at 891,000 tons annually) a radical change in the process of manufacturing steel has greatly affected the quality of basic slag. Its chemical composition is extremely variable, depending on the chemical constituents of the iron ores and the materials used in the methods of purification, and largely on the process of making steel. In the Bessemer process, phosphoric acid formed combines with the lime used, and the compound of calcium phosphate is removed in the slag, while in the Siemen's open hearth process (now in use) "iron-ore and lime are charged on to a basic hearth heated by producer gas, and the molten metal poured over the heated lime and ore. ** The slag formed in the basic open hearth process is much greater in volume, and there is a corresponding decrease in phosphoric acid content compared with the basic Bessemer process." ¹ Then again, in the process of eliminating sulphur from the metal, fluorspar is added which contributes largely to the reduction of the citric-solubility of the slag. The investigations of the Department have, therefore, been directed to three problems with regard to basic slags, one is to find the

¹ Robertson, G. S., "Basic Slags and Rock Phosphates," Cambridge University Press, 1922.

relationship between the chemical compositions of different slags and their manurial values, the second is to study their citric solubility, and the third is to determine the soil conditions on which their relative efficiency depends. As its value rests largely on the form in which phosphorus is present in the compounds, Gimingham is carrying on analytical research in this direction, especially in connection with the determination of fluorine.

I have dwelt at length on this research item with a view to impress on our agricultural investigators the indispensable need of phosphatic manures to modern farming. In bone manures, we have an abundant source of phosphorus, the annual export of which exceeds 100,000 metric tons; and yet most of our soils are deficient in available phosphorus content. Leather's analytical work and the recent Soil Survey in certain parts of India left no room for doubt that one of the underlying causes of the gradual impoverishment of the Indian soils is the phosphorus-hunger. The problems of utilizing bones for agriculture must receive an immediate attention from the popular Government, even if it becomes necessary to put an embargo on the ever-increasing export trade. While the financiers should come forward to organise the industry of manufacturing bone phosphates in India, the agricultural investigators should be carrying on *systematic* experiments to find out the conditions under which phosphatic manures yield best results.

The line of research with potassium fertilisers being carried on by Eden is interesting. The extensive potassium deposits in Stassfurt (Germany) are the principal source whence this fertiliser is obtained. About 90 per cent. of the annual output are now used for agriculture both in Germany and abroad. But, owing to variations and irregularities in different strata of the deposits, salts of various compositions are obtained. The

chief deposits yielding potassium are Carnallite ($\text{KCl} \cdot \text{MgCl}_2 \cdot 6\text{H}_2\text{O}$), Kainite ($\text{K}_2\text{SO}_4 \cdot \text{MgSO}_4 \cdot \text{MgCl}_2 \cdot 6\text{H}_2\text{O}$) and Sylvanite ($\text{K}_2\text{SO}_4 \cdot \text{MgSO}_4 \cdot \text{KCl} \cdot \text{MgCl}_2 \cdot \text{NaCl} \cdot 6\text{H}_2\text{O}$). The salts that come from the Alsatian deposits contain chlorides.

Now, chlorides are known to be injurious to plant growth. Therefore, while investigating the relative manurial value of various potassium fertilisers, special attention is given to the action of the chlorine ion. No conclusive data can be given at this initial stage of the experiments, but the summary of results so far obtained may be of interest. With low-grade potash deposits, such as sylvanite the yield is lower, which is possibly due to the excess of chlorine ion present as sodium chloride. The presence of chlorine ion in excess also greatly affects the quality of the crop, such as potato; but with high-grade salts only a little difference is seen. It is to be remembered that the action of these artificial manurial substances varies considerably in relation to divergent soil conditions. Therefore, the relative effects of different fertilisers on soils are also being studied, following the methods suggested by Parker¹ and Ischerekov.²

In the analytical work of determining potash, the perchlorate method is found to be unreliable owing to the presence of chloric acid. Page has taken up the problem, and attempts are being made to devise such methods in which deflagration set in by the addition of alcohol and also the formation of compounds due to the presence of chloric acid may be eliminated.

Side by side with the problem of increased crop production, agriculturists are faced with an aspect of

¹ Parker, F. W. "Methods of studying the Concentration and Composition of the Soil Solution." *Soil Science*, Vol. XII, pp. 209-232, 1921.

² Ischerekov, V. 1907. "Obtaining the Soil Solution in an Unaltered Condition."

consumption which bears an intimate relation to production. I mean for an agriculturist it is not now merely the question of increasing the crop yield, but also of satisfying the consumer's demand for quality. The industries which depend on agricultural crops have developed a need which must be met; with the development of modern civilisation and improvements in transportation facilities, the capacities and tastes of consumers have also undergone a great change. Therefore, the things the consumer wants, and is willing to pay for, are the things farmers must grow. And yet nothing is definitely known as to the conditions under which a desired quality of crop could be obtained. So the Rothamsted invites the consumers' representatives, (such as wholesale dealers, manufacturers, etc.,) to visit the numerous experimental plots and to pronounce their judgment as regards the quality of crops grown under different treatments. This enables Rothamsted to advise farmers definitely what must be grown and under what conditions.

At the same time the chemists are at work to discover the factors that appear to contribute to the improvement of the quality of crops.

The chemists from the Institute of Brewing, the chief consumers of barley, are carrying on analytical work to detect the influence of manuring and of topographical and climatic factors on the malting quality of barley. Barley-malt is an essential ingredient of beer; so much so that it is described as "the soul of beer." As the chemical composition of barley varies widely, attempts are being made to study its precise relation with malting-efficiency.

In India the brewing industry is still in its infancy, total production not being more than 8,000,000 gallons per annum, although the barley export from India during the last few years is valued at £1,500,000 per annum.

Sorghum (known as "chulam") which is the staple grain crop of Southern India, is now considered to be a possible substitute for barley for malting operations.¹

We have so far dealt with the chemical fertiliser investigations, but the agricultural practice all over the world indicates that farmyard manure is, on the whole, more effective than any other single chemical substance. The problems associated with its application however, are not simple; and therefore, we turn to the Department of Fermentation, where Richards has been carrying out very instructive researches.

¹ Norris. "Sorghum as a Substitute for Malting Barley," Jour. Agric. India, June, 1923.

II.

The Rothamsted Experimental Station.

(Department of Fermentation.)

The conservation of nitrogen as a means of adequate food-production is one of the fundamental problems of agriculture, and it is this very problem that has been persistently investigated by the Rothamsted Experiment Station since the day of its foundation. The loss of nitrogen from the rich cultivated soil, which could not be satisfactorily accounted for, has been a subject of patient investigation for many years. But the elucidation of its causes is rendered extremely complex owing to the presence of so many factors in the soil; and therefore, it was thought that the investigation of the changes that take place in a manure heap might afford some precise information. It should also be noted that the impetus from the situation of the war-emergency undoubtedly gave a direction to the researches on the loss of nitrogen. The Rothamsted Experiment Station, through the generous assistance of the Hon. Rupert Guinness, now Lord Elveden, secured the services of Richards, who has had a wide experience as a sewage chemist, and then this much-neglected investigation of farmyard manure was seriously taken in hand.

From the point of view of Indian agriculture, this line of research is extremely instructive. The Indian cultivator fully recognises the value of farmyard manure; the idea of cowdung being 'sacred' may have probably originated from its utility as an effective crop-producing manure. Among the numerous agricultural proverbs one finds occasional references to cowdung as being "Lakshmi," the symbol of the Goddess of Wealth. And

Science, VI.



SIR JOSEPH HENRY GILBERT
(First Chemist at Rothamsted)

yet, owing to the shortage of fuel, the bulk of it is burnt, and whatever is left for agricultural purposes loses a considerable amount of fertilizing ingredients through mismanagement of the manure heap.

A few remarks as to the composition of farmyard manure may be necessary for clear understanding of the research problem I am about to describe here.

Farmyard manure consists of more or less decomposed materials of the excreta of animals mixed with straw or some such litter used in order to absorb urine and moisture of the solid excreta. However important a place the chemical fertilizers may occupy in the economy of scientific agricultural practice, they cannot altogether replace the farmyard manure or *vice versa*. In fact, one is not the substitute of the other. While the chemical fertilisers supply certain essential plant foods, the farmyard manure, besides adding nitrogen and potash to soils, brings about certain physical and biochemical conditions which contribute largely to the maintenance of soil fertility. Its value is also enhanced by its residual effect on succeeding crops.

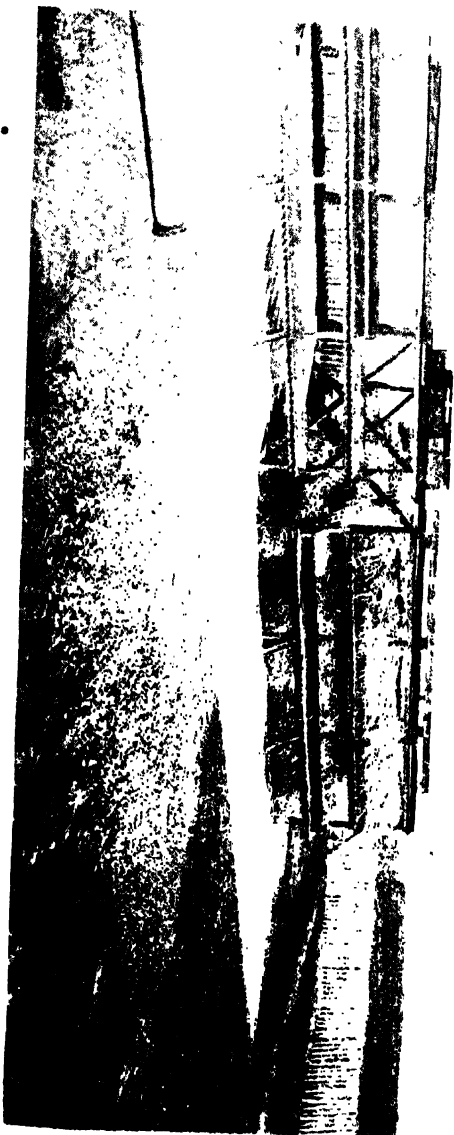
During the war, one of the most important questions which arrested the attention of the Government, was to conserve all possible sources of manures so necessary for increased food-production in the country, and naturally the question of economical use of the most popular fertilizing material, farmyard manure, which suffers serious loss through fermentation and leaching, was raised. It was estimated ¹ that 37,000,000 tons of farmyard manure are made per annum in the United Kingdom valued at £11,000,000; but the loss in making and storing the manure heap, according to Russell, is more than £5,000,000 each year.

¹ Russell and Richards. On Making and Storing of Farmyard Manure. (Journal of Royal Agric. Soc., Vol. 77.)

This loss of nitrogen however, cannot altogether be avoided. Animal excreta which are partially decomposed materials, in combination with straw, come under the influence of micro-organisms, and thus a series of complex biochemical changes are brought about. But the loss of nitrogen from the manure heap can certainly be reduced, if the mechanism of such fermentive changes could be ascertained. We know that the fertilizing value of farmyard manure depends largely on the actions of micro-organisms; and yet they are also partly responsible for the loss of nitrogen, the volatilization of ammonia and leaching being other factors. Therefore, the aim of the agriculturist is to augment that type of fermentive process in which the micro-organic activities may go on without involving much loss to this important constituent of the manure heap.

The chemistry of farmyard manure is exceedingly variable and complex; its composition depends on the nutritive value of the food-stuffs and on the extremely complicated physiological process of digestion. That there is an intimate relation between animal food and dung was first shown by Boussingault¹, but not until necessary physiological data with regard to mechanism of digestion was available "the residual manurial value of various feeding stuffs" could be determined. Here again the lead was taken by Lawes and Gilbert. From 1854 onwards they carried on several investigations with a view to establishing a definite relation between the nutritive constituents of the animal foods and the distribution of the various fertilizing ingredients, such as nitrogen, phosphorus, and potash, of dung and urine. The science of animal physiology has made such a great advance during the last two decades that almost a precise relation

¹ Boussingault. *Economie Rurale*, 2nd Ed., Vol. 1, p. 1. Quoted by Russell in (3).



The Straw Filter.

between food-stuffs and manure may now be established. Instead of going into any detailed discussion of physiological investigations, I shall be content to reproduce the following summary made by Russell and Richards:¹

“1. Only small amounts of the nitrogen, phosphorus and potash in the food are retained by the animal; he lives mainly on the carbon and hydrogen of the food, and passes out the bulk of the manurial constituents into his excretions.

2. A good deal of the nitrogen is taken up by the animal, but not kept; it passes out in the urine and becomes a rich plant food.

3. The more digestible the food, the greater is the amount of nitrogen thus taken up and excreted, *i. e.*, converted into plant food.

4. There is no loss of nitrogen, potash or phosphoric acid in the animal, and an exact balance sheet can be made up. If the amount and the composition of the food is known, the amount and composition of the excretions can be calculated by simple arithmetic.”

But as in actual practice it is not possible to collect fæces and urine separately, the calculations do not follow the simple arithmetic. Then again, the use of absorbents, such as straw, dry leaves, etc., introduces a new constituent in the manure heap.

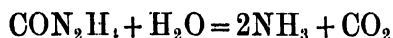
Thus, we find several conditions affecting the value of farmyard manure. Besides the kind of animal, its age, its chief function, and the quality and quantity of the foodstuff, there are such important factors as the relative proportions of solid and liquid excreta, the quality and quantity of litter used and the manner in which the manure heap is treated. It is to this question of treatment the most serious consideration has to be given.

¹ Russell and Richards. On making and Storing of Farmyard Manure. (Journal of Royal Agric Soc., Vol. 77).

Through a series of carefully conducted analyses and field trials, it is shown that manure retains its fertilizing constituents best, when it is kept as compact as possible under sufficiently moist conditions protected from exposure to atmospheric influences. But in spite of these precautions, the loss of nitrogen could not be obviated. Russell and Richards¹ designed a series of laboratory experiments to find out what actually happens in manure when subjected to aerobic and anaerobic conditions.

Although the phenomenon of fermentation is of common occurrence, the mechanism of its process is still shrouded with marvellous complexities of multiple factors which are brought into play by micro-organisms. Therefore, in the whole field of biological chemistry the subject of fermentation is one of the most fascinating items of research.

Fermentation is a natural process known to be accelerated under favourable conditions by aerobic and anaerobic bacteria, the former acting in the presence of air and the latter in its absence. Aerobic bacteria attack the nitrogenous substances evolving ammonia in the process, which, in addition to carbon-dioxide, is partly converted into ammonium carbonate. The reaction may be stated as follows :



The bacterial decomposition of protein has all the appearance of hydrolytic change. In fact, the hydrolysis of proteins into amino-acid groups is the first stage of putrefaction ; but further degradation of amino-acids brought about by bacterial activities may take place in two ways. Either an amino-group may be eliminated

¹ Russell and Richards: Changes taking place during the storage of farmyard manure. (*Journal of Agric. Soc.*, Vol. VIII., Part IV. 1917).

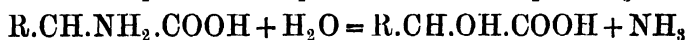
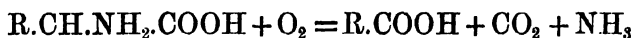
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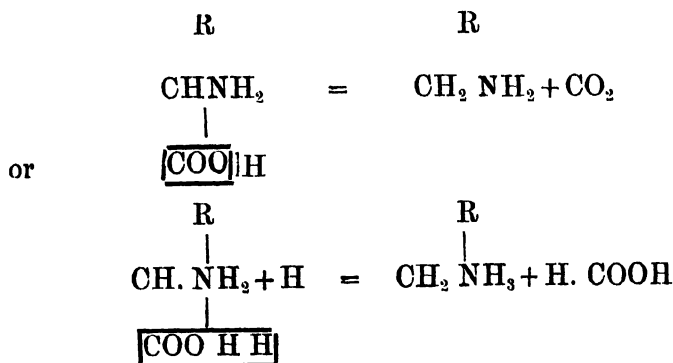
Moistening the Straw-heap.

(deaminization) or a carboxyl-group may be removed (decarboxylation). It must not be understood that the processes work only within two alternatives. Various modifications and combinations of these processes are possible, depending chiefly on micro-organic activities.

De-aminization occurs under both aerobic and anaerobic conditions. Researches on protein metabolism¹ have thrown much light on the fundamental character of the process of de-aminization, which seems to be essential for synthetic activity. It is a normal function of both moulds and the micro-organisms of putrefaction. De-aminization by bacteria has been much investigated by Brasch and Neuberg² and we know that aerobic or anaerobic conditions, profoundly affect the molecular groupings of various products. Dakin³ describes the reaction under aerobic processes thus :—



Decarboxylation may take place either by the simple removal of CO_2 or the Carboxyl-group (*e.g.*, formic acid) in which case there must be a reduction process.



¹ Cathcart. *Physiology of Protein Metabolism*. (Longman's Biochemical Monographs, 1912.)

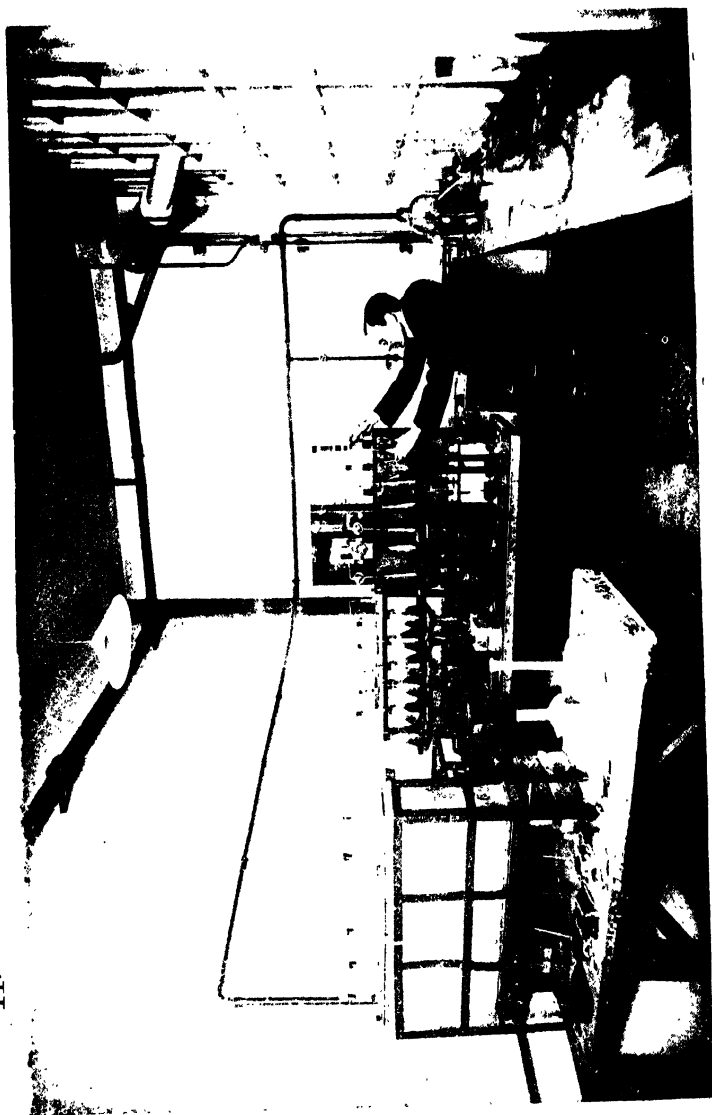
² Brasch and Neuberg. *Biochem. Zeit.*, 1908, Vol. 13, p. 299, and 1909, Vol. 22, p. 408. Quoted in (3).

³ Dakin. *Journal of Biological Chem*, 1908, Vol. 4, p. 63. Quoted in (3).

It is found that decarboxylation takes place under an anaerobic process producing amines among the regular products of putrefaction. The fundamental destruction brought about by these two processes under aerobic and anaerobic conditions must be noted. While under aerobic conditions ammonia splits off, leaving a complex acid, both this or the splitting off carbon dioxide leaving a complex amine may take place under anaerobic conditions. We thus see that with the change in the aerobic conditions the products change also.

No precise information is yet available as to conditions which determine the occurrence of deaminization or decarboxylation. In the bacterial decomposition of complex nitrogenous compounds present in soils and manure, it is likely that under normal conditions both the processes go on simultaneously, and as aerobic activity commences soon, possibly deaminization preponderates. Russell and Richards' researches on the changes that occur in the storage of farmyard manure have not only thrown much light on the problems related to conservation and utilization of farmyard manure and allied materials, but have also furnished some useful guiding hypotheses as regards the complexities brought about in nitrogen-carbon cycles through the processes of fermentation. From the theoretical discussions outlined above, it must not be concluded that the investigators are unanimously agreed upon the probable reactions of nitrogen-carbon cycles in the soil or manure. The accumulations of such data as obtained by Russell and Richards serve as links, the systematic connections of which may some day reveal the true nature of the cycles.

From their investigations it is indicated that the fundamental nature of fermenting complex in manure, strictly under aerobic and anaerobic conditions, closely resembles that detected either in the bacterial decomposition



The Laboratory of the Department of Fermentation.

of protein or in sewage purification. In all cases the complex nitrogen compounds probably break down in the same way, involving the formation first of amino-acids and then of ammonia. But in manure heap ammonia does not accumulate, indicating thereby the possibilities of the occurrence of further reactions.

The aerobic and anaerobic experiments were made side by side and the changes in nitrogen compounds were found to be different, as will be seen from the graphic illustration (Fig. 1). Comparing these results with those observed in the anaerobic decomposition of sewage or in the breaking down of protein, one is led to believe that whatever destruction may appear in manure heap is solely due to the conditions of storage.

Now, the chief point to remember in connection with the distinctive features of aerobic and anaerobic storage is that there is a loss of nitrogen in the former case. Although the decomposition of complex nitrogen compounds goes on more vigorously in the presence of air than under anaerobic conditions, it is found that there is no loss of nitrogen under wholly aerobic conditions. What are then, the chief factors that bring about the loss of nitrogen, and in what form does it occur? The usual explanation of the loss being wholly due to the volatilization of ammonia proved to be insufficient and so also the idea that soluble nitrogen compounds may be lost by leaching; for the loss occurs also in sheltered heap. Therefore, Russell and Richards, assuming that the missing nitrogen may be escaping as gas, set up a very ingenious experiment, maintaining, as nearly as possible, such conditions of decomposition as are found in manure heap. As the apparatus they used may be useful to the students for nitrogen investigations, I reproduce

here the diagram (Fig. 2) as well as the description of the technique¹ they employed.

"A weighed quantity of farmyard manure of known composition was placed in the flask A, connected to an acid bulb B (to absorb ammonia), an alkali bulb (to absorb CO_2)² and a soda lime tube D, and to the mercury pump E, whereby a stream of air could be drawn through. The pump was so arranged that the whole of the air was delivered back into the flask. The system was closed and it was made rigidly air-tight and absolutely beyond any possibility of leakage by building the apparatus up with the aid of the blowpipe and making all the joints of blown glass, and further by having mercury seals on the tap and the flask. A manometer F was attached so that changes in pressure could be read: any absorption or evolution of gas could thus be measured. As oxygen was absorbed in quantity it was essential that a pure supply should be available, free not only from sulphur dioxide, carbon dioxide, etc., but also from traces of nitrogen: this was ensured by sealing on to the apparatus an electrolytic vessel G, charged with baryta, by the electrolysis of which very pure oxygen was obtained. The volume of oxygen was determined by measuring the volume of hydrogen in the nitrometer H.

The manure having been put into the flask A, and the last joints blown, the pump was started working and sufficient air was extracted to allow the auxiliary mercury lifting pump to come into action. The system was then completely closed, the circulation commenced, and the pressure on the manometer read.

The volume of the apparatus now had to be determined. This was done by letting in a known volume of oxygen

¹ Russel and Richards: Changes taking place during the storage of farmyard manure. (*Journal of Agric. Soc.* Vol. VIII. Parr. IV. 1917.)

² The glass spiral between B and C is to relieve the strain on the apparatus.

II.



The mixed culture of *Azotobacter* and *B. lactis arogenes*.

from the electrolytic vessel and measuring the drop in pressure thus caused.

Knowing the volume of the apparatus and the percentage composition of the air contained therein, it was easy to calculate the volume of nitrogen initially present.

The circulation was then kept up for several hours a day, the mercury being lifted by the auxiliary water-pump so that the operation was automatic. The acid in the one set of bulbs caught any trace of ammonia carried over, while the alkali in the other set took out the CO_2 ; the movements of the manometer showed the net change in pressure. Fresh oxygen was periodically admitted from the electrolytic vessel.

An occasional sample of gas was pumped out, measured and analysed and the manometer was read. Thus the composition of the air inside the apparatus could be known whenever desired.

When the experiment had continued long enough the gas was pumped out, measured and analysed. This gave the volume of nitrogen finally present.

Lastly the manure was collected from the flask, weighed and analysed, and the acid from the bulbs was also distilled to collect the ammonia, which was added to that in the manure."

Thus, it was possible to detect the change in amount of nitrogen in the manure and also in volume. That there is an evolution of nitrogen during the decomposition of manure, and that only under partial aerobic conditions loss of nitrogen occurs, may be concluded from the experiment.

As regards the cause of the evolution of nitrogen, there are many hypotheses advanced by the European investigators; but without entering into a detailed account of them, it may be stated that in view of the experimental evidences of the loss of nitrogen only under the partial

anaerobic and aerobic conditions, the theory of either a direct oxidation or a direct reduction process is disproved. Russell and Richards favour the alternate nitrification and denitrification hypothesis, and find explanations of the changes in molecular groupings which become stable or unstable depending on aerobic or anaerobic conditions. They suppose that under anaerobic conditions molecular groupings tend to rise, which become unstable as soon as aerobic conditions set in, and *vice versa*, thus splitting off nitrogen in the process of decomposition. Such changes occur in non-nitrogenous compounds, as for example¹ P-hydroxy-phenyl-propionic acid formed by the de-aminization of tyrosine is oxidised to p-hydroxy-phenyl-acetic acid. A similar shortening of the carbon chain is brought about by oxidation in the production of succinic acid from glutamic acid by putrefaction. It is supposed that some complex nitrogen compound could act likewise by eliminating nitrogen. In a manure heap the conditions between aerobic and anaerobic fluctuate, giving rise to a succession of a series of decompositions, and thus evolving in reactions gaseous nitrogen.

Be that as it may, we now know from Russell and Richards' investigations that "the objects to aim at in a manure heap must be to secure (a) as much dry matter (b) as much ammonia and (c) as little loss of nitrogen as possible," and only under complete anaerobic conditions and at a temperature of 26°C, such an ideal situation may be obtained. It is obvious that the complete anaerobic conditions cannot easily be attained, but what has been successfully done for silage may also be accomplished for storing farmyard manure. Anaerobic fermentation produces marsh gas and hydrogen; and as the former is

¹ Barger. *Simpler Natural Bases*. (Longman's Monographs on Biochemistry, 1914.)

combustible, there is certain amount of risk in subjecting farmyard manure to a complete anaerobic process. For the present, the best alternative is to keep manure well compacted and sheltered, bearing in mind the several factors that govern the fundamental changes taking place during the storage. The Department of Fermentation is continuing the investigation with a view to devising a method whereby the perfect anaerobic conditions may be attained without any risk from combustible gas.

As the success of experiments depends on the careful attention given to details, I wish to summarise, for our students, an abstract of the methods employed by the authors.¹ In the first place great care should be taken in obtaining a representative sample. The experimental heap should be built up with small quantities of dung taken out from the bulk heap of fresh manure, and from each forkful two small samples should be laid aside for analysis. These samples should be thoroughly mixed by passing them through a large meat-mincing machine. In the second place, the laboratory analyses should all be made in duplicate thus obtaining four values for each item. For nitrogen determination, the Kjeldahl-Gunning method is sufficiently accurate; the ammonia may be estimated as follows:—Add 500 c. c. of distilled water and about three grams of magnesia to 10 grams of the sample manure, and see if the solution is alkaline to litmus. Distil it at ordinary pressure, and titrate.

For the amide nitrogen, digest 10 grams of the manure sample with 200 c. c. of 10% sulphuric acid for 10 hours on the water bath, making just alkaline with caustic soda.

¹ Russel and Richards: Changes taking place during the storage of farmyard manure. (*Journal of Agric. Soc.* Vol. VIII, Part. IV, 1910).

Distil. From the value thus obtained, deduct the ammonia figure.

From the problem of the storage of farmyard manure we now proceed to consider what Richards and his associates have done to explore the possibilities of utilising another source of nitrogen namely, the sewage sludge.

The need of increasing the output of nitrogenous manures during the war revived in the United Kingdom the past efforts to find the best means of utilizing sewage sludge, the manurial value of which, according to Russell¹ is estimated at £18,000,000 per annum, if the total excrements of the inhabitants of the country could be applied to the land. It is a well-known fact that "the farmers of forty-centuries" in China and Japan conserve every scrap of human excreta for manuring purposes, and Fowler² asserts that the fundamental economic stability of those countries depends on this universal practice. Professor F. H. King gives very interesting statistics³ of the estimated quantity of manure applied annually in Japan. Japan cultivates less than 14 million acres (in 1910) of land, to which are applied annually about 24 million tons of human excreta, 23 million tons of compost made from animal droppings and waste materials mixed with grass, straw, etc., 5 million tons of green weeds carefully harvested from uncultivated regions, and 776,000 tons of ashes. These manures make an average annual application of 3·8 tons per acre containing, according to the accepted analyses

¹ Russell. Work of the Rothamsted Experimental Station from 1914 to 1919. (Journal of the Board of Agriculture, Vol. XXVI, No. 5.)

² Fowler. Conservation of Nitrogen with special reference to Activated Sludge. (Journal of Ind. Inst. of Science, Vol. III, 227, 279, 1920.)

³ King. Farmers of Forty Centuries. (1911, Madison, Wis.) Abstract in the Orange Judd Farmer (Jan. 22, 1910).

II

FIRST CROP : BARLEY.



No Manure.

Activated
Sludge.

Slate-bed
Sludge.

Nitrate of
Soda.

Increase of dry
matter in crop.

...

65 p.c.

22 p.c.

51 p.c.

of official Japanese chemists, 54 pounds of nitrogen, 14.8 pounds of phosphorus and 29.2 pounds of potassium. Since the last decade Japan has also been importing a large quantity of chemical fertilizers and oil-cakes to make up any deficiency in the complete plant-foods. Sir Humphrey Davy, writing as early as 1827 in his treatise, 'Agricultural Chemistry,' states as follows :

"The Chinese, who have more practical knowledge of the use and application of manures than any other people existing, mix their night-soil with one third of its weight of a fat marl, make it into cakes, and dry it by exposure to the sun. These cakes, we are informed by the French Missionaries, have no disagreeable smell, and form a common article of commerce of the Empire."

The value of sewage as a source of nitrogen has long been appreciated by agricultural chemists, but the numerous problems associated with sewage disposal need to be solved, before its use can be universally recommended. In the first place, the offensive accompaniments involved in its use have to be got rid of, bearing in mind the important question of sanitation ; and in the second place, the loss of nitrogen has to be guarded against, bringing under control, as far as possible, the fermentive processes that take place both in the solid and liquid substances.

Various methods of sewage treatment have been suggested, but one worked out by Dr. Gilbert J. Fowler in Manchester, is considered practicable and efficient, and has received much attention from agricultural investigators. It has been found that if air is passed through sewage, a number of changes take place, eventually throwing down a brown deposit, with the progress of oxidation, which is known as the 'activated' sludge.

The process as carried out in practice is described by Fowler¹ as follows :—

“The screened sewage passes into a long narrow *aeration* tank into which air is forced in a state of fine division. It has been found that this fine division is necessary for the sake of economy of air, and it is effected by the use of what are known as *diffusers* of porous material through which the air is forced, creating a fine emulsion of air and sewage. The effluent passing away at the end of the aeration tank is purified, but contains, of course, activated sludge in suspension which must be separated and returned to the inlet of the tank. The aeration tank is, therefore, followed by a *settlement* tank in which the activated sludge rapidly settles out, and from which the clear and purified effluent passes. The deposited activated sludge can be lifted from the bottom of the settlement tank by means of compressed air either back to the inlet of the aeration tank or out of the system altogether on to sludge drying beds, from which it can be removed and used as manure.”

There are several types of sludges produced by different methods, but to what extent the activated sludge process of sewage purification may be successful as a practical proposition, probably needs further confirmatory evidences. However, the fertilizing value of sludge thus obtained has been carefully tested both by field and pot-culture experiments. At the Rothamsted Experiment Station, Brenchley and Richards,² working with slate-bed and activated sludges, which are produced under aerobic conditions, and the precipitation and septic-tank sludges formed under anaerobic conditions, found that the

¹ Fowler. Conservation of Nitrogen with special reference to activated sludge. (Journal of Ind. Inst. of Science, Vol. III, No. 227, 279, 1920.)

² Brenchley and Richards. Fertilising Value of Sewage Sludge. (Journal of Chem. Industry, Vol. 39, p. 177T, 1920.)

SECOND CROP : MUSTARD.



No Manure.	Nitrate of Soda.
...	478 p.c.
Activated Sludge.	16 p.c.
Increase of dry matter in crop.	
105 p.c.	

manurial value of the former sludges was distinctly greater than the others. The chemical analyses of sludges used in the experiments shows a wide difference in nitrogen contents.

<i>Sludge.</i>		<i>Activated.</i>	<i>Slate-bed.</i>	<i>Precipitation</i>
		%	%	%
Moisture	7.26	55.65	37.67
Organic Matter	...	76.72	20.76	24.81
Inorganic Matter	...	16.02	23.59	37.52

Calculated on the sludges dried at 100°C.

Nitrogen	7.09	2.63	0.89
Phosphoric Acid	...	3.82	0.34	0.66
Potash	1.12	0.08	0.07

The pot-culture experiments made with (a) activated and (b) slate-bed sludges indicate the superiority of the former not only as a high-yielding fertilizer, but also as a manure which has a considerable residual value. Brenchley and Richards carried out these experiments with an equivalent dressing of nitrate of soda, thus introducing a standard for comparison. Their experiments also bring out the interesting fact that the crop increases "are roughly proportional" to the availability of its nitrogen content as determined in the nitrification experiments made with sludges. The photographs of the pots, kindly lent by the Department, show the difference in plant growth under the respective treatments.

Prior to the above test, Arden¹ carried out a series of field experiments near the Withington sewage works with a view to determine the value of activated sludge as a source of available nitrogen for growing crops, and his

¹ Arden. Activated-sludge Process of Sewage Purification. (Journal of Chem. Industry, Vol. 39, p. 60, 5, 1920.)

results are in accordance with those obtained at Rothamsted.

But further questions were raised as regards the behaviour of the complex nitrogen-carbon cycle in the activated sludge : what is the source of its high nitrogen content ; is the nitrogen recovery in the case of activated sludge more than any other method of sewage purification, and does any loss of nitrogen occur in the sludge, and if so, under what conditions ? With a view to obtain certain experimental evidences for answering some of these questions, Richards and Sawyer¹ undertook a series of investigations at the Rothamsted Laboratories, and the financial support came from the Ministry of Agriculture.

They attempted to present a comprehensive outline of the mechanism of the nitrogen-carbon cycle that operates, as in the case of farmyard manure, chiefly through the micro-organic population of the sludge. Contrary to the views held by certain investigators, their experiments adduce no evidences of fixation of atmospheric nitrogen by activated sludge, but indicate that increased content of nitrogen is derived from the ammonia of sewage, which in addition to colloidal nitrogen, is removed from the sewage. If activated sludge is aerated for a short time in an ammoniacal solution, the recovery of nitrogen is quantitative, but under continuous aeration, loss of nitrogen occurs.

The distinctive character of Richard and Sawyer's experiments lies in the fact that they extended their observations to the micro-organic population of the activated sludge more thoroughly than the previous workers. For, in all the processes known as ammonification, nitrification and denitrification, the activities of the different

¹ Richards and Sawyer. Further Experiments with Activated Sludge. (*Journal of Chem. Industry*, Vol. 41, p. 62T, 1922.)



No
Manure
...

Activated Sludge.
940 p.c.

Slate-bed
Increase of dry matter
in crop Sludge.
79 p.c.

Nitrate of
Soda
473 p.c.

groups of micro-organisms under various conditions have much to do with the fate of nitrogen. The ammonia-fixing organisms are strongly aerobic, and they flourish in the presence of carbohydrates. The activated sludge under aeration, therefore, presents a suitable medium for the activity of these organisms with the result that a large proportion of sewage ammonia is 'fixed' in the sludge.

I need not repeat here the part played by the nitrifying and denitrifying bacteria in the activated sludge. Fundamentally, their function is the same as in the case of the manure heap, but a reference should be made to the existence of a third group of organisms—protozoa—whose development is known to proceed "simultaneously with the other biological changes." The correlation between the bacterial numbers and the total active protozoa has lately been investigated by the Protozoological Laboratory at Rothamsted, and this discovery has thrown much light on the complex biological changes that take place in soil, sludge or farmyard manure. I shall refer to this work in detail in a subsequent paper, but the interesting observation of Richards and Sawyer with regard to the source of *extra* nitrogen in the activated sludge may be quoted here.

"If aeration is very good," the authors state, "large ciliates predominate; under less aerobic treatment flagellates are mostly found. By devouring the ammonia-assimilating bacteria, the protozoa transfer a considerable part of the nitrogen gained from the sewage into the protein of their own cell structure." It has been estimated that the numbers of protozoa in well-activated sludge run up to 1,000,000 per gram of wet sludge. In order to determine to what extent they influence the process of sewage purification, a number of experiments with partial sterilisation of activated sludge were conducted; and it was found that the rapid increase in bacterial

numbers following the antiseptic treatment produced no improvement in the purification of sewage, but seriously retarded the normal fermentation processes by suppressing nitrifying organisms. Again, through the suppression of the protozoa population, the value of activated sludge as a fertilizer may also be depreciated.

Although the claim of activated sludge as having a high manurial value has been established by field trials, there remain two problems of sewage disposal—one is to find out economical methods of drying the sludge produced, and the other is to devise means of recovering nitrogen now lost in effluent—which must be solved if sewage is to be utilized for agricultural purposes.

His long association with researches on the fermentation processes, and his wide experience in matters of sewage disposal enabled Richards to suggest a method of sewage purification by which the highly dilute ammoniacal compounds largely derived from urine and liable to be lost in effluent, may be recovered and made available as a fertilizer. We shall discuss at length the process he has employed in the making of the synthetic straw manure. It is this readily fermentible carbohydrate, straw, which he used as a filtering medium for sewage purification. The idea must have occurred to Richards while carrying out, in collaboration with Hutchinson, researches on the fermentive changes of nitrogen and carbon. For, as far as the natural requirements for the growth of the organisms are concerned, they may be furnished by the combination of straw with an excess of carbon, and sewage with an excess of nitrogen, in their respective constituents. Richards states that "it took from 10 to 14 days for the straw to be thoroughly activated. The optimum temperature was about 35°C, but at 60°F to 70°F the process was successful in 10 days.

I

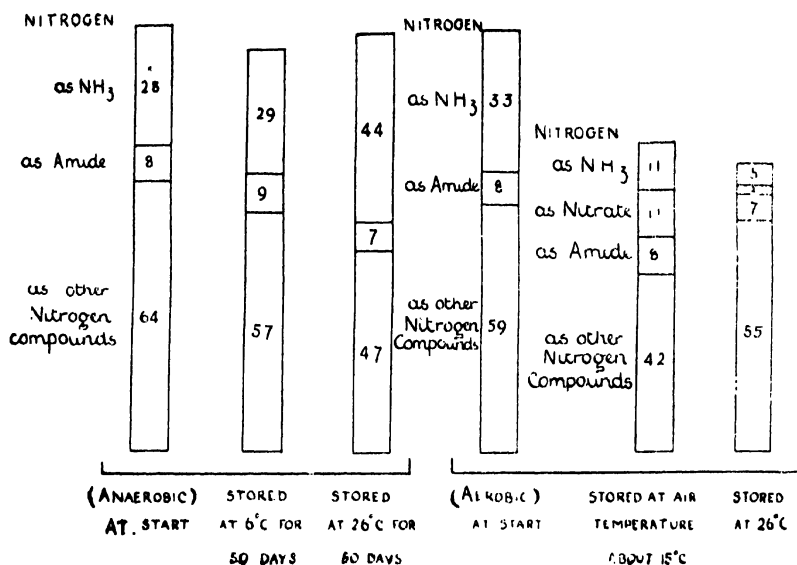


FIG. 1

Changes in Nitrogen Compounds in Farmyard manure (Bullock manure) stored in the Laboratory under anaerobic and aerobic conditions at different temperatures. (*Russell and Richards.*)

(From the Journal of the Agricultural Science, Vol. VIII, 1916-17.)

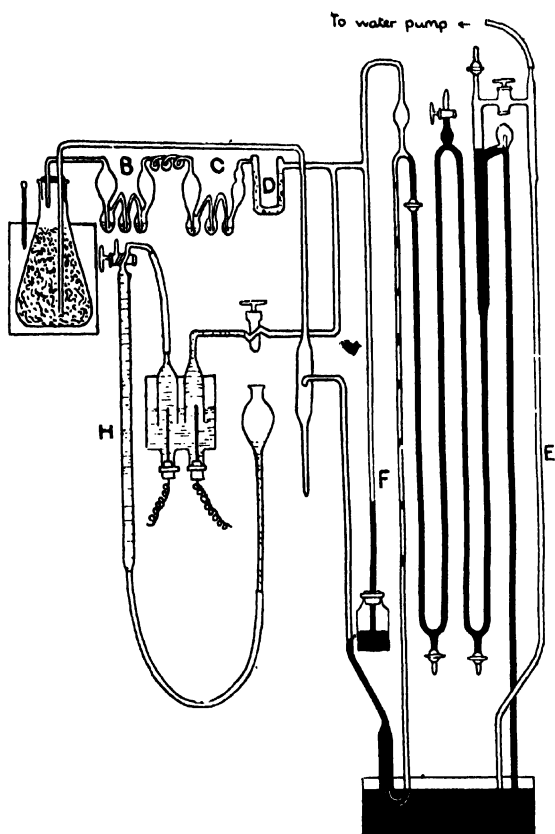


FIG. 2

Apparatus to ascertain absorption or evolution of gaseous nitrogen
from decomposed farmyard manure.

(*Russell and Richards.*)

(*From the Journal of Agricultural Science, Vol. VIII.*)

At the end of three weeks the straw was loaded with as much nitrogen as it would take up."

The laboratory experiments, showing that straw saturated with a dilute solution of ammonium carbonate (10 parts of N per 100,000) gradually becomes active in removing nitrogen from the solution, led to the construction of a straw-filter (see Photograph) which is described by Richards and Weeks as follows:—¹

"An angle-iron framework, 20 feet long, 10 feet wide, and 7 feet 6 inches high, was fixed over a sloping concrete floor, with two main horizontal members, one 2 feet and the other 4 feet 6 inches above the floor, passing along each side. On these members 1 inch diameter galvanised iron pipes, 12 feet long, were laid, spaced at 4 inches apart, centre to centre; the length of the pipes allowed them to project 12 inches beyond the framework at each end to give hand-hold for drawing.

Transverse horizontal angle-irons were fixed across the bed 3 feet above the upper of the two floors so formed, to tie the verticals together and also to carry the distributing troughs. Two V-shaped wooden troughs were laid over the filter, the sides having notches about 12 inches apart with a nail driven under each to act as a drip point."

The rate of pumping has to be regulated; for instance, if it is rapid, there would be a very small amount of nitrogen fixation; and when it is reduced to, say, 96 gallons per cubic yard per day, there was a decided improvement in the nitrogen recovery.

Richards gives the results of analyses of samples from his straw filter which clearly show the gain of nitrogen in straw-removed filter.

¹ Richards and Weekes. Reprints from the Proceedings of the Engineering Conference, 1921, of the Institute of Civil Engineers.

				<i>Parts per 1,00,000</i>	
				Sewage.	Effluent.
Ammoniacal Nitrogen	18.92	10.19	
Oxygen absorbed in 4 hours from perman- ganate	31.88	20.82	
Dissolved oxygen taken up in 5 days	80.4	14.9	
Chlorine	31.0	39.3	
				Dry matter %	
Total nitrogen in raw straw	0.50		
Total nitrogen in straw removed from filter	1.62		
Total nitrogen after storage for 6 months			2.06		

The amount of nitrogen recovery in the above straw-filter process should encourage agricultural investigators in India to explore this source of nitrogen from sewage, and the Municipalities and District Boards should seriously consider the suitability of this method of sewage-purification. Fowler, whose contributions to the questions of sewage disposal both from agricultural and sanitary points of view are well known, is now in India as the Chief of the Indian Institute of Science (Bangalore). He is already, I understand, directing his students to such research problems, and it may be hoped that in India some cheaper vegetable waste products, other than straw could be utilised for sewage-filter. Of course, the details of decomposition and fermentive processes under the tropical conditions have to be studied first, and if any such method proves to be successful it would, besides conserving a source of nitrogen, greatly improve the sanitary conditions, especially of Indian villages.

The genesis of the idea of purification of sewage by straw-filter may probably be traced to the experiments which have been going on at Rothamsted for some time with a view to making synthetic straw manure, subjecting

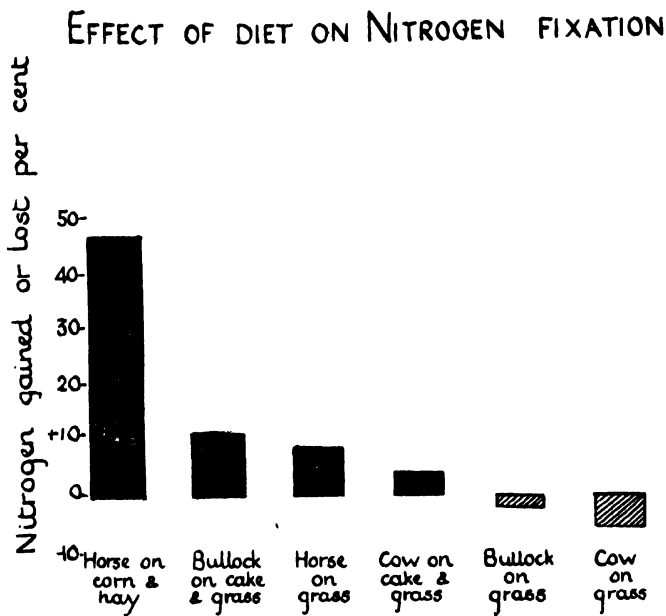


FIG. 3

(After Richards)

(From the Journal of Agricultural Science, Vol. VIII, 1916-17.)

straw to bacterial decomposition. Now, straw consists of 5-10% starch, 20-40% pentosan and the rest cellulose of various types, but its nitrogen compounds are inert and therefore do not contribute anything to the supply of energy to organisms. The efforts to secure a more or less complete breakdown in fermenting straw were not successful, partly due to the lack of sufficient information as to the food supply of the cellulose-decomposing organisms, and partly due to the characteristic resistance of such substances as pectin, lignin, etc., to any fermentive process. However, Richards' investigations on the processes of fermentations furnished him with some interesting and suggestive data. As for example, while dealing with the problem of conversion of ammonia into complex nitrogen compounds, during the fermentation of urine, straw, and faeces, he found that only in the manure heap composed entirely of horse manure, there was any increase in the complex nitrogen compounds. His further experiments¹ show that horse faeces fixed nitrogen under aerobic conditions in the presence of sufficient moisture and calcium carbonate, and the fixation is a function of the food-stuffs. I reproduce here the diagrammatical representation of "the effect of diet on the nitrogen-fixing power of faeces from the horse, bullock and cow." (*See fig. 3.*)

Looking for the organism or organisms which may take part in the nitrogen fixation, Richards could obtain evidences to show that it is brought about by "a mixed culture (see the microphoto) of *Azotobacter* and *B. lactis aerogenes*. Of these the latter is normally present in faeces; *Azotobacter* is not, but readily infects faeces. Both organisms

¹ Richards. Fixation of Nitrogen in Faeces. (*Journ. of Agric. Science*, Vol. VIII, Part 3, 1917.)

are present in the soil used and will fix nitrogen in raw fæces but not in sterile fæces." This fact suggests that there must be some actual or potential food present in fæces for organisms which fix nitrogen.

The problem of making synthetic manure from straw resolves itself into two fundamental questions, one is to find the cellulose-decomposing organisms which could attack straw under aerobic conditions, and the other is to provide food for these organisms in the form of a nitrogen compound.

Now, since Omelianski's¹ investigations into cellulose-fermentations under anaerobic conditions, a good deal of the attention of the biochemists has been directed to the problems related to the processes involved. Although his observations were confined to anaerobic fermentations, it was suspected that there must also be some cellulose-fermenting microbes which work in the presence of air under favourable conditions. Researches of Van Iterson² first indicated the nature of the aerobic fermentation of cellulose. He found that the presence of nitrates accelerated the process, but was not able to isolate the pure culture of aerobic cellulose-decomposing organisms. Hutchinson and Clayton³ working in the Rothamsted Laboratory could, however, obtain an essentially aerobic organism, named *Spirochaeta cytophaga*, whose carbon requirements appear to be satisfied only by the easily decomposable carbohydrate materials of cellulose. The investigators found that any other carbonaceous compounds, especially those containing

¹ Omelianski. Abstract published in Journal Chem. Soc., 1902, 2. p. 468. Originalsee. Compt. rend. 1895, Vol. 121, 653-655; 1897, Vol. 125, 970-973, 1131-1133.

² Van Iterson. Cont. Bakt. Par. II, 1904, Vol. II, 689-698.

³ Hutchinson and Clayton. On the Decomposition of Cellulose by an Aerobic Organism. (Journ. Agric. Soc., Vol. IX, 2, 1910.)

reducing groups, were not only unfavourable to its growth, but they exerted an inhibitive action on the process of decomposition. As regards its nitrogen requirements, simple compounds such as ammonium salts, nitrates, etc., appear to be quite suitable. They found *Spirochaeta cytophaga* present in soil and "capable of breaking down cellulose with comparative ease," provided the organism could utilise some form of nitrogen for its growth. This discovery indicated the way of "making a nitrogenous and humus-forming manure from straw by bacterial decomposition," and Richards in collaboration with Hutchinson continued his investigations as regards the necessary conditions for straw decomposition. The following conclusions are drawn from their experiments.¹

(1) Air-supply.—Typical rotting occurs only under aerobic conditions.

(2) Supply of soluble nitrogen compounds in suitable concentration. The concentration of even the weakest undiluted urine is above the maximum limit for decomposition. No rotting occurs until the concentration of ammonium carbonate has been sufficiently reduced by volatilization. The rate of decomposition is accelerated under a neutral or slightly alkaline reaction.

(3) The rate of fermentation is rapid at a temperature near 35°C.

The success of the process, then, depends on both physical and chemical factors which are found to induce fermentation. The straw heap should be uniformly moist, and well-aerated, and the amount of nitrogen compounds to be used is based on the calculations of the inter-relations between nitrogen and straw. According to Hutchinson

¹ Hutchinson and Richards. Utilisation of Straw and the Production of Artificial Farmyard Manure. (Journ Min. Agric., XXVIII, 5, 1921.)

and Richards, the amount varies between 0.70 and 0.75 parts of nitrogen per 100 parts of dry straw, and within these limits fermentation goes on without any appreciable loss of nitrogen. To guard against acid conditions, which may arise through organic fermentations, finely ground chalk or limestone is sprinkled over the straw heap. The uniform alkalinity is to be aimed at.

Theoretically speaking, the method for the preparation of artificial manure is a simple one, but many practical difficulties have to be overcome in order to satisfy all the essential requirements of straw fermentation. However, as soon as its possibilities as an efficient manure have been clearly indicated both by laboratory experiments and field trials, the process has been covered by Letters Patent and the problem of its commercial development is now in the hands of the Agricultural Developments Company, Limited.

As it is essential that the process should be carried out precisely on the lines established by experiments, the Company has prepared a chemical mixture which supplies nitrogen up to the limit of required concentration. Amoore, who has been working out the practical difficulties of the method says, "the uniform saturation of the straw with water is the difficult part of the job." Already the Company has designed such transportable devices by which they are capable of turning out 2,000 tons of straw manure per annum. It is estimated that one ton of dry straw makes four tons of fresh and about three tons of rotted manure, but it remains to be seen to what extent farmers avail themselves of the process.

Through the courtesy of the Company I was able to obtain certain experimental data,¹ showing the

¹ Adco's publication. Process for breaking down Straw, 1923.

fertilising value of synthetic straw manure, which are cited below.

Comparison between Synthetic Straw Manure and Ordinary Farmyard Manure.

	Per cent Moisture.	Organic Matter.	Nitrogen.	Phosphate.	Potash.
Bullock dung ...	74.65	78.06	2.45	1.04	2.83
Cow dung ...	80.56	78.55	2.20	0.99	2.24
Synthetic straw manure ...	76.00	78.60	2.16	0.56	0.60

Comparative field trials on heavy loam at Rothamsted and on light soil at Working indicated that this synthetic straw manure was as good as high quality farmyard manure.

FIELD TRIALS.

Working Farm. (Light soil.)

	Weight applied per acre.	Yield per acre in tons.	
		Potatoes (1920).	Mustard (1921).
Synthetic straw manure ...	20 tons	6.92	2.91
Sulphate of ammonia ...	4 cwt.	5.32	2.37
Dung ...	20 tons	6.18	4.66
No manure	3.44	1.83

Rothamsted. (Heavy loam.)

	Weight applied per acre.	Tons per acre. Potatoes.
Synthetic straw manure ...	16 tons	2.42
Sulphate of ammonia ...	4 cwt.	1.57
Dung ...	16 tons	2.98
No manure	1.46

The most significant increases of potato yield over 'no manure' were obtained at Heath Place farm under the Company's supervision, as shown by the following data :—

<i>Increases over no manure.</i>			Potatoes (1922).	
Chemical fertilisers only	4 Tons	1 Cwt.
„ „ and cow manure	7 „	2 „
„ „ and synthetic straw manure...	5 „	16 „

The synthetic straw manure behaves almost as efficiently as the farmyard manure, both in seasonal and residual manurial value.

I have dwelt at length on this question of making synthetic straw manure and have quoted experimental evidences to prove its claim as an equally efficient substitute; for, the fundamental problem before Indian agriculture is the supply of nitrogenous manures, preferably those which add organic materials to soils. While, owing to the scarcity of fodder in India, straw cannot be utilised for the purpose, there are numerous refuse organic materials such as cotton stalks, cane waste, mahua from distilleries, dry leaves, and dried water-hyacinth, etc., which may be successfully fermented with some modifications of the process Richards employs. It is likely that some of these organic materials may prove to be better media for the cellulose-decomposing organisms than the straw. The analysis¹ of cane waste shows that it contains 0.24% N, 0.108% CaO, 0.25% K₂O, and 0.142% P₂O₅, and the quantity available annually is estimated at 22 millions of tons. I trust that the success Richards has had with the straw decomposition, will encourage the Departments of Agriculture and

¹ Adinarayan, Rao K. A New Source of Manure. (Journal of Agriculture of India, Vol. XVII, Part 5, p. 476.)

the Research Institutes in India to investigate into the possibilities of utilising such abundant sources of waste organic materials for the making of efficient fertilisers.

From the researches described in these pages we may realise how intimate is the relation between the problem of plant-nutrition and the micro-organic population of the soil. Here, indeed, lies the unexplored field of research. We shall next acquaint ourselves with the investigations that are in progress at Rothamsted on soil-bacteria, protozoa, algæ and other micro-organisms.

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